Superfluids Under Rotation
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TWISTED VORTEX STATE

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Content

1) Does the twisted state exist at all?

Hydrodynamic theory
- uniform twist
- linear theory of nonuniform twist

2) Generation of twisted vortex states

Numerical simulations

3) Observation in superfluid $^3$He

sketch of twisted vortex lines
Twisted vortex states in classical fluids

Figure: http://www.amc.edu.au/research/areas/cavitation/projects/

Stability of a polygon of helical vortices (Okulov 2004)
Rotating superfluid

Equilibrium vortex state

Twist at the ends

Twisted vortex state


→ no mention of twisted vortices!

Taylor-Proudman theorem: "Any slow motion in rotating fluid is columnar"
1) Does the twisted state exist?

Hydrodynamic equations

Superfluid velocity $v_s$

\[ \nabla \times v_s = 0 \quad \text{except at vortex lines} \]
\[ \nabla \cdot v_s = 0 \]

$\Rightarrow$ Vortex lines fully determine $v_s(r, t)$.

Line velocity $v_L$

\[ v_L = v_s \]

Add mutual friction

\[ v_L = v_s + \alpha \hat{l} \times (v_n - v_s) - \alpha' \hat{l} \times [\hat{l} \times (v_n - v_s)] \quad (1) \]
Continuum model of vorticity

(Hall and Vinen 1956)

\[ v_s = \langle v_s^{\text{local}} \rangle \]

\[ \nabla \times v_s = \omega \]

\[ \nabla \cdot v_s = 0 \]

Line velocity \( v_L \)

\[ v_L = \tilde{v}_s + \alpha \hat{l} \times (\tilde{v}_n - v_s) - \alpha' \hat{l} \times [\hat{l} \times (v_n - \tilde{v}_s)] \]  \( (2) \)

where \( \tilde{v}_s = v_s + \nu \nabla \times \tilde{\omega} \), \( \nu = (\kappa/4\pi) \ln(b/a) \).

Alternatively, one can use equation of motion for \( v_s \):

\[ \frac{\partial v_s}{\partial t} = v_s \times \omega + \nu(\omega \cdot \nabla)\tilde{\omega} + \nabla \phi \]
Uniformly twisted vortex state

Most symmetric state [cylindrical coordinates \((r, \phi, z)\)]

\[ \mathbf{v}_s = v_\phi(r) \hat{\phi} + v_z(r) \hat{z}, \]

⇒ vorticity

\[ \mathbf{\omega} = \frac{1}{2} \nabla \times \mathbf{v}_s = \frac{1}{2} \left[ -v'_z \hat{\phi} + \left( \frac{v_\phi}{r} + v'_\phi \right) \hat{z} \right] \]

Calculate vortex line velocity from (2). For a stationary state the radial velocity must vanish. This implies

\[ (\Omega r - v_\phi) \left( \frac{v_\phi}{r} + \frac{dv_\phi}{dr} \right) - v_z \frac{dv_z}{dr} + \frac{\nu}{|\mathbf{\omega}| r} \left( \frac{dv_z}{dr} \right)^2 = 0. \]

This implies that the helical vortices rotate together with the normal fluid, \( \mathbf{v}_L = \mathbf{v}_n = \Omega \times \mathbf{r} \).

⇒ There exists a family of stationary, uniformly twisted states.
In a finite cylinder the total axial current must vanish,

\[ \int_0^R dr \, r v_z = 0. \tag{3} \]

The functions \( v_z(r) \), \( v_\phi(r) \) and the radial displacement of the vortices compared to equilibrium state, \( \epsilon(r) \), are sketched in the figure.

The simplest case is helical vortices with a wave vector \( Q(r) = \text{constant} \). This has

\[ v_\phi(r) = \frac{(\Omega + Q v_0) r}{1 + Q^2 r^2}, \]
\[ v_z(r) = \frac{v_0 - Q \Omega r^2}{1 + Q^2 r^2}. \tag{4} \]
Linearized hydrodynamics

Assume general velocity with circular symmetry

\[ \mathbf{v}_s = v_r(r, z, t)\hat{r} + v_\phi(r, z, t)\hat{\phi} + v_z(r, z, t)\hat{z} \]

Assume small deviation from rotating equilibrium.
⇒ waves of the form

\[ v_r = ckJ_1(\beta r) \exp(ikz - i\sigma t) \]
\[ v_z = ic\beta J_0(\beta r) \exp(ikz - i\sigma t) \]

Dispersion relation [Glaberson, Johnson and Ostermeier (1974), Henderson and Barenghi (2004)]

\[ \frac{\sigma}{\Omega} = \frac{-i\alpha(\beta^2 + 2k^2\eta_2) \pm i\sqrt{\alpha^2\beta^4 - 4(1 - \alpha')^2k^2(\beta^2 + k^2)\eta_1\eta_2}}{\beta^2 + k^2} \]

where \( \eta_1 = 1 + \nu k^2 / 2\Omega \) and \( \eta_2 = 1 + \nu(\beta^2 + k^2) / 2\Omega \).

In order to understand the dispersion, we study special cases.
1) $\beta \to 0$, corresponds to a short cylinder

$\Rightarrow$ 2 Kelvin wave modes (Hall 1958)

$$k_{\pm} = i \sqrt{\frac{2\Omega \pm \sigma}{\nu}}$$

and an inertial mode

$$k_i = 0$$

At low frequency ($\sigma \ll \Omega$) these give just the columnar motion because Kelvin waves are evanescent. No twisted state.
2) $k \to 0$, corresponds to a long cylinder

$\Rightarrow$ 2 modes

The point $k = \sigma = 0$ corresponds to uniform twist!

At finite $k$ the twist obeys diffusion equation

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial z^2}, \quad D = \frac{1}{d} \left( \frac{2\Omega}{\beta^2} + \nu \right)$$

(5)

where $f(z, t) = v_r$ or $v_z$. 
Summary of two opposite limits

Parallel plates
- columnar vortices

Long cylinder
- twisted vortices
2) Generation of twisted vortex states

- superfluid in a cylinder
- cylinder rotating at $\Omega > \Omega_c$, but
no vortices in the initial state
- generate vortices at one place

- vortices propagate along the cylinder and
- vortex ends rotate around the cylinder axis
Why vortex ends rotate?

Normal component rotates at \( v_n = \Omega \times r \).

Superfluid component: vortex lines move with the average superfluid velocity

1) vortex state: \( v_s \approx \Omega \times r \)
\( \Rightarrow \) vortex lattice rotates at angular velocity \( \Omega \)

2) no vortices: \( v_s = 0 \)

3) vortex front
average superfluid angular velocity \( \Omega/2 \) \( \Rightarrow \) vortex ends rotate at angular velocity \( \Omega/2 \)
\( \Rightarrow \) propagating vortex ends lag behind
Numerical simulation

Vortex line velocity (2)

\[ v_L = v_s + \alpha' \hat{l} \times [(v_n - v_s) \times \hat{l}] + \alpha \hat{l} \times (v_n - v_s). \]

\(v_s\) is calculated from Biot-Savart integral. (Risto Hänninen)

The front and the twisted state is confirmed by numerical calculation movie
Axial velocity

\[ r = \frac{R}{6} \]

Azimuthal velocity

\[ \frac{v_z}{\Omega R} = \frac{v_\phi - \Omega r}{\Omega R} \]

movie
Main observations
- the twisted state has axial current.
- individual vortices become unstable to generate Kelvin waves at large axial current
- the vortices glide at the bottom plate
⇒ relaxation of the twist
- the relaxation is determined by the diffusion equation.
3) Experiment in superfluid $^3$He-B

Vortex state was generated as discussed above.

The axial velocity $v_z$ affects the texture, which is seen by NMR.
vortex injection

counterflow peak absorption

$\Omega = 1.50 \text{ rad/s}$
$R = 3 \text{ mm}$
$T = 0.55 T_c$
$p = 29 \text{ bar}$

Larmor absorption

$\Omega = 1.45 \text{ rad/s}$
$N = 0$

$\Omega = 1.50 \text{ rad/s}$
$N = N_{eq}$

counterflow peak absorption

frequency shift (kHz)

time (s)
\( \Omega = 1.4 - 1.6 \text{ rad/s} \)

\( p = 29 \text{ bar} \)

Diffusion constant

\[
D = \frac{1}{d} \left( \frac{2\Omega}{\beta^2} + \nu \right) \propto \frac{1}{\text{mutual friction constant}}
\]  

(6)
Conclusions

Twisted vortex state is a possible state in long rotating cylinders.

The twisted state can be generated by vortex injection.

The twisted state has been seen in superfluid $^3$He-B.