# Introduction to the Theory of Superconductivity and Superfluidity

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## Abstract

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### Contents

# I. TWO-FLUID DESCRIPTION OF SUPERFLUIDITY

Superfluidity is an ability of a fluid to flow without friction through narrow tubes. We shall see why one needs a narrow tube in this definition. The most striking realizations of superfluidity are thermo-mechanical effects. Some of them are ilustrated in Figs. 1 and 2

## A. Landau criterion

One can understand the absence of dissipation in a quantum liquid from the following arguments. Assume that our fluid is at zero temperature in a ground state. Since a quantum system cannot change its energy continuously, the system has to create an excitation to absorb the dissipated energy. Let the energy of an excitation in the stationary fluid be $\epsilon(\mathbf{p})$ with a momentum $\mathbf{p}$. For a fluid moving with a velocity $\mathbf{v}$, the energy of the excitation in the laboratory frame is

$$\epsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}$$

It is favorable to create such an excitation if

$$\epsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v} < 0$$

This inequality can be satisfied if

$$|\mathbf{v}| > \frac{\epsilon(\mathbf{p})}{|\mathbf{p}|}$$

or if at least

$$|\mathbf{v}| > \min \left\{ \frac{\epsilon(\mathbf{p})}{p} \right\} \tag{1}$$

Excitations are created if this condition is satisfied. If the minimum is nonzero

$$v_c \equiv \min \left\{ \frac{\epsilon(\mathbf{p})}{p} \right\} \neq 0 \tag{2}$$

no excitations can be created when the flow has a velocity below $v_c$: The quantum fluid has no dissipation if it flows with a velocity $v < v_c$. This is called the Landau criterion of superfluidity.

FIGURES

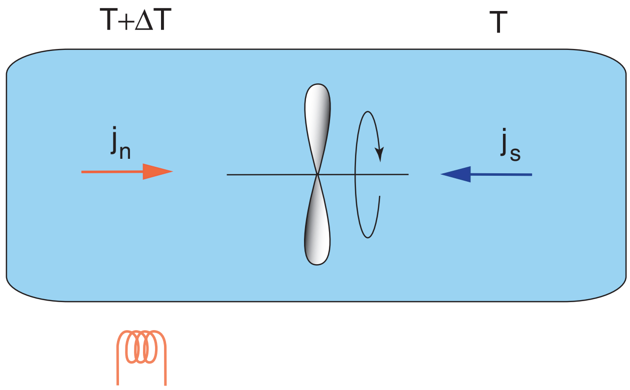


FIG. 1. The fan rotates if one end of the closed tube is heated.

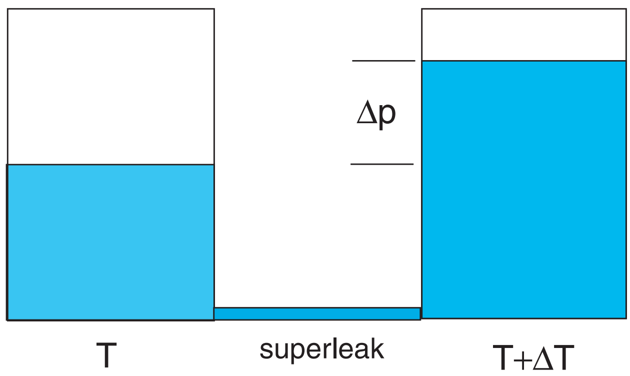


FIG. 2. A pressure head is established between two vessels kept at different temperatures and connected by a superleak.

For a finite temperatures, there are already thermally created excitations which, if they move, experience friction through their interactions with the container walls. At the same time, for flows with $v < v_c$ there is a part of fluid that moves without dissipation. In fact, one can present the total mass current (momentum) per unit volume of the fluid as a sum of two parts

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s \tag{3}$$

The velocity $\mathbf{v}_n$ is associated with the motion of excitations, $\rho_n$ is the coefficient of proportionality with the dimension of the mass density. The motion of excitations produces friction as in a usual or "normal" fluid. Thus $\mathbf{v}_n$ and $\rho_n$ are called the normal velocity and density of the fluid, respectively. The velocity $\mathbf{v}_s$ is associated with the motion without friction; it is called the superfluid velocity, while the proportionality coefficient $\rho_s$ is the superfluid density. If the entire fluid moves as a whole with a velocity $\mathbf{v}_s = \mathbf{v}_n = \mathbf{v}$, the mass current is

$$\mathbf{j} = (\rho_n + \rho_s)\,\mathbf{v} = \rho\mathbf{v}$$

where $\rho$ is the total density of the fluid. Therefore

$$\rho = \rho_n + \rho_s \tag{4}$$

### B. Two-fluid hydrodynamics

Since the superfluid motion below $v_c$ does not create excitations, it does not transfer momentum to and thus it does not exert a force on an external object immersed into fluid. This means that the superfluid motion is a potential flow, characterized by the zero-vorticity condition

$$\operatorname{curl}\mathbf{v}_s = 0 \tag{5}$$

Therefore, the full time derivative (the material time derivative) can be presented as a gradient

$$\frac{d\mathbf{v}_s}{dt} = \frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\,\mathbf{v}_s = -\nabla \mu_s \tag{6}$$

where $\mu_s$ is the potential (per particle mass) that acts on the superfluid component. Therefore, $\mu_s$ has the meaning of the chemical potential of the superfluid part of the liquid. Using Eq. (5) we obtain

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla\left(\mu_s + \frac{\mathbf{v}_s^2}{2}\right) = 0 \tag{7}$$

Consider a reference frame $K_0$ that moves with the velocity $\mathbf{v}_s$ with respect to the laboratory frame $K$. We have in the laboratory frame (per unit volume)

$$\mathbf{j} = \rho \mathbf{v}_s + \mathbf{j}_0 \tag{8}$$

$$E = \frac{\rho \mathbf{v}_s^2}{2} + \mathbf{v}_s \mathbf{j}_0 + E_0 \tag{9}$$

Here $E_0$ and $\mathbf{j}_0$ are the energy and current in the moving frame which depend on the fluid velocity $\mathbf{v}_n - \mathbf{v}_s$ in the moving frame. The energy obeys the thermodynamic identity

$$dE_0 = T\,dS + \mu\,d\rho + (\mathbf{v}_n - \mathbf{v}_s) \cdot d\mathbf{j}_0 \tag{10}$$

Here $\mu$ is the chemical potential (defined per particle mass), $S$ is the entropy per unit volume. The current density is

$$\mathbf{j}_0 = \rho_n (\mathbf{v}_n - \mathbf{v}_s)$$

which, together with Eq. (8), indeed reproduces Eq. (3).

To find the thermodynamic relation for the chemical potential we consider a system of $N$ particles with an energy $\tilde{E}$ in a volume $\tilde{V}$ and with a momentum $\tilde{\mathbf{J}}$. Its energy obeys

$$d\tilde{E} = T\,d\tilde{S} - p\,d\tilde{V} + \tilde{\mu}\,dN + \mathbf{v} \cdot d\tilde{\mathbf{J}}$$

The chemical potential per particle $\tilde{\mu}$ is defined as

$$\tilde{\mu} N = \tilde{E} - T\tilde{S} + p\tilde{V} - \mathbf{v} \cdot \tilde{\mathbf{J}}$$

Calculating the variation of the both sides of this equation and using the identity for the energy we find

$$N d\tilde{\mu} = -\tilde{S}\,dT + \tilde{V}\,dp - \tilde{\mathbf{J}} \cdot d\mathbf{v}$$

or

$$\begin{aligned} d\tilde{\mu} &= -\frac{\tilde{S}}{\tilde{V}}\frac{\tilde{V}}{N}\,dT + \frac{\tilde{V}}{N}\,dp - \frac{\tilde{V}}{N}\frac{\tilde{\mathbf{J}}}{\tilde{V}} \cdot d\mathbf{v} \\ &= -S\frac{\tilde{V}}{N}\,dT + \frac{\tilde{V}}{N}\,dp - \frac{\tilde{V}}{N}\mathbf{j} \cdot d\mathbf{v} \end{aligned}$$

Dividing this by the particle mass we get

$$d\mu = -\frac{S}{\rho}\, dT + \rho^{-1}\, dp - \rho^{-1}\mathbf{j}\cdot d\mathbf{v}$$

where $\rho = mN/\tilde{V}$ and $\mu = \tilde{\mu}/m$.

Therefore, in our case, the chemical potential (per unit particle mass) satisfies the thermodynamic relation

$$d\mu = -\sigma\, dT + \rho^{-1}\, dp - \rho^{-1}\mathbf{j}_0\cdot d\left(\mathbf{v}_n - \mathbf{v}_s\right) \tag{11}$$

where $\sigma = S/\rho$ is the entropy per unit particle mass.

Eq. (11) helps to identify the function $\mu_s$ in Eq. (7) as the chemical potential of the superfluid particles. Indeed, if we apply Eq. (11) to superfluid component alone we should put the entropy to zero since the superfluid part of the fluid is all in one (the ground) state. Moreover, we also put $\mathbf{j}_0 = 0$ since there is no relative motion in the superfluid part. As a result, Eq. (11) for superfluid component gives

$$\nabla\mu_s = \rho_s^{-1}\,\nabla p$$

At the same time, according to the Euler equation,

$$\rho_s \frac{d\mathbf{v}_s}{dt} = -\nabla p$$

Comparing these two equations we arrive at Eq. (7).

In equilibrium, the chemical potentials of all the components of the fluid should be equal, therefore, $\mu_s = \mu$, and

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla\left(\mu + \frac{\mathbf{v}_s^2}{2}\right) = 0 \tag{12}$$

In addition, the flow obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0 \tag{13}$$

Since the entropy of the superfluid part is zero, the entropy of the fluid is only associated with the normal component (excitations). Therefore, the continuity equation for entropy is

$$\frac{\partial S}{\partial t} + \mathrm{div}\ (S\mathbf{v}_n) = W_S \tag{14}$$

Here $W_S$ is the dissipation function that describes the entropy production due to viscosity of the normal component.

The momentum conservation takes the form

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial r_k} = 0 \tag{15}$$

where $\Pi_{ik}$ is the tensor of the momentum flow. In the absence of the normal viscosity, it is

$$\Pi_{ik} = \rho_n v_{n\,i} v_{n\,k} + \rho_s v_{s\,i} v_{s\,k} + p\delta_{ik} \tag{16}$$

The two-fluid hydrodynamic equations (12) – (15) are quite complicated because $\rho_n$, $\sigma$, $\mu$, etc., depend on $\mathbf{v}_n - \mathbf{v}_s$ and cannot be calculated without a microscopic theory. However, there are situations where some progress can be made already on the basis of these general equations.

### C. First and second sounds

Assume that the fluid is at rest on average, and consider small variations of its density and velocity. Within the first approximation in $\mathbf{v}_s$ and $\mathbf{v}_n$ we obtain from Eqs. (12) – (15)

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu = 0 \tag{17}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\mathbf{j} = 0 \tag{18}$$

$$\frac{\partial (\rho\sigma)}{\partial t} + (\rho\sigma)\,\mathrm{div}\,\mathbf{v}_n = 0 \tag{19}$$

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla p = 0 \tag{20}$$

We assume also that system is in thermal equilibrium with $\omega\tau \ll 1$ so that the dissipation is absent.

Eqs. (18), (20) give

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p \tag{21}$$

Eq. (19) yields

$$\frac{\partial^2(\rho\sigma)}{\partial t^2} + (\rho\sigma)\,\mathrm{div}\,\frac{\partial \mathbf{v}_n}{\partial t} = 0 \tag{22}$$

while Eqs. (17), (20) result in

$$\mathrm{div}\,\frac{\partial \mathbf{v}_s}{\partial t} + \nabla^2\mu = 0$$
$$\rho_n \mathrm{div}\,\frac{\partial \mathbf{v}_n}{\partial t} + \rho_s \mathrm{div}\,\frac{\partial \mathbf{v}_s}{\partial t} + \nabla^2 p = 0$$

which gives

$$\mathrm{div}\,\frac{\partial \mathbf{v}_n}{\partial t} = (\rho_s/\rho_n)\,\nabla^2\mu - \rho_n^{-1}\nabla^2 p$$

Inserting this into Eq. (22) we obtain

$$\frac{\rho_n}{\rho\sigma}\frac{\partial^2(\rho\sigma)}{\partial t^2} + \rho_s\nabla^2\mu - \nabla^2 p = 0 \tag{23}$$

We express here $\partial^2\rho/\partial t^2$ through Eq. (21) writing to the first approximation in small variations

$$\partial^2(\rho\sigma)/\partial t^2 = \sigma\partial^2\rho/\partial t^2 + \rho\partial^2\sigma/\partial t^2$$

We neglect the term $(\partial\sigma/\partial t)(\partial\rho/\partial t)$ which is of the second order in variations. Next we use the thermodynamic relation Eq. (11) whence

$$\nabla^2\mu = -\sigma\,\nabla^2 T + \rho^{-1}\,\nabla^2 p \tag{24}$$

As a result,

$$\frac{\partial^2\sigma}{\partial t^2} = \frac{\rho_s\sigma^2}{\rho_n}\nabla^2 T \tag{25}$$

Equations (21) and (25) determine the behavior of thermodynamic variables in a sound wave.

Denote $\delta p$ and $\delta T$ variations of pressure and temperature in the sound wave. Since

$$\delta\rho = \frac{\partial\rho}{\partial p}\delta p + \frac{\partial\rho}{\partial T}\delta T$$
$$\delta\sigma = \frac{\partial\sigma}{\partial p}\delta p + \frac{\partial\sigma}{\partial T}\delta T$$

we obtain from Eqs. (21) and (25)

$$\frac{\partial\rho}{\partial p}\frac{\partial^2\delta p}{\partial t^2} + \frac{\partial\rho}{\partial T}\frac{\partial^2\delta T}{\partial t^2} - \nabla^2\delta p = 0 \tag{26}$$
$$\frac{\partial\sigma}{\partial p}\frac{\partial^2\delta p}{\partial t^2} + \frac{\partial\sigma}{\partial T}\frac{\partial^2\delta T}{\partial t^2} - \frac{\rho_s\sigma^2}{\rho_n}\nabla^2\delta T = 0 \tag{27}$$

We look for a solution in the form of a plane wave

$$\delta p, \delta T \propto \exp\left[-i\omega\left(t - x/u\right)\right] \tag{28}$$

The equations yield

$$\left(\frac{\partial\rho}{\partial p}u^2 - 1\right)\delta p + \frac{\partial\rho}{\partial T}u^2\delta T = 0 \tag{29}$$
$$\frac{\partial\sigma}{\partial p}u^2\delta p + \left(\frac{\partial\sigma}{\partial T}u^2 - \frac{\rho_s\sigma^2}{\rho_n}\right)\delta T = 0 \tag{30}$$

These equations can be simplified if we neglect a small thermal expansion and put

$$\frac{\partial\rho}{\partial T} = \frac{\partial\sigma}{\partial p} = 0$$

Equations (29), (30) then have two solutions:

$$u_1 = \sqrt{\frac{\partial p}{\partial\rho}} \quad \text{while} \quad \delta T = 0 \tag{31}$$

and

$$u_2 = \sqrt{\frac{\sigma^2\rho_s}{\rho_n(\partial\sigma/\partial T)}} \quad \text{while} \quad \delta p = 0 \tag{32}$$

Equation (31) describes the usual (first) sound, i.e., variation of density and pressure at a constant temperature. Note that we neglected the difference between adiabatic and isothermal processes when we put the thermal expansion to zero. Equation (32) describes variations of temperature and entropy. The pressure does not change. Since the thermal expansion is absent, the density is also constant. This type of perturbation is called the second sound.

### D. Vortices in a rotating superfluid

In a rotating container, the normal component rotates with the vessel such that its velocity is $\mathbf{v}_n = \mathbf{\Omega} \times \mathbf{r}$. Note that this flow field has $\mathrm{curl}\, \mathbf{v}_n = 2\mathbf{\Omega}$. Due to the potentiality of superflow Eq. (5) the superfluid component cannot rotate with a rotating container; it should be stationary in the laboratory frame. However, this fact contradicts to the experiment which shows that the entire fluid rotates with the same angular velocity $\mathbf{\Omega}$. For example, if the superfluid did not rotate, the meniscus height would contain an extra factor $\rho_n/\rho$ as compared to the normal fluid (see Problem 3).

Consider the apparent contradiction in more detail. The condition of a potential flow of the superfluid component Eq. (5) implies that the superfluid velocity can be presented as a gradient of a flow potential

$$\mathbf{v}_s = \nabla \Phi \tag{33}$$

This equations yields identically that $\mathrm{curl}\, \mathbf{v}_s = 0$ everywhere in the fluid, as it should be. However, if it were correct absolutely at any point in the fluid, the superfluid could not rotate. Is it possible to "spoil" Eq. (5) in such a way that the damage to it would be minimal? One can argue as follows. Assume that Eq. (5) is violated only at separate points in the fluid while the rest of the superfluid remains curl-free. To have a nonzero contribution to the bulk of the fluid we should assume a $\delta$-function-type of the violation. This $\delta$-function should depend only on the coordinates $\mathbf{r}_\perp$ in the plane perpendicular to the rotation axis since $\mathbf{v}_s$ should be uniform along the rotation axis. In particular,

$$\mathrm{curl}\, \mathbf{v}_s = A\hat{\mathbf{\Omega}}\delta^{(2)}\left(\mathbf{r}_\perp - \mathbf{r}_{0\perp}\right) \tag{34}$$

where $\hat{\mathbf{\Omega}}$ is the unit vector along the rotation axis and $A$ is a coefficient to be determined. The coordinate $\mathbf{r}_{0\perp}$ determines thus a singular point on a plane where the curl-free condition of superflow is violated. In three dimensions, we thus obtain a singular line.

According to the Stokes theorem, Eq. (5) can be written as

$$\oint \mathbf{v}_s \, d\mathbf{r} = 0$$

where the contour integral is taken along any contour in the fluid. However, if now we take the contour that encircles the above singular line, the result is

$$\oint \mathbf{v}_s \, d\mathbf{r} = \int \operatorname{curl} \mathbf{v}_s \, d\mathbf{S} = A$$

where the second integral is taken over the area confined within the contour. Again, the contour around the singular line can be taken arbitrary because $\operatorname{curl} \mathbf{v}_s = 0$ everywhere outside the line. Multiplying the above equation by the particle mass $m$ we have

$$\oint m\mathbf{v}_s \, d\mathbf{r} = mA$$

According to the general principles of the quantum mechanics (Bohr–Sommerfeld quantization) the above integral should be quantized. In other words, one should have $mA = 2\pi\hbar n$ where $n$ is an integer. Therefore,

$$\oint \mathbf{v}_s \, d\mathbf{r} = \frac{2\pi\hbar n}{m} \equiv \kappa_0 n \tag{35}$$

or

$$\operatorname{curl} \mathbf{v}_s = \kappa_0 n \mathbf{e}_L \, \delta^{(2)} \left( \mathbf{r}_\perp - \mathbf{r}_{0\perp} \right) \tag{36}$$

where $\mathbf{e}_L$ is the unit vector along the line.

This generalizes Eq. (5): For $n = 0$ one has the curl-free condition (5). Nevertheless there can be singular lines such that the circulation of superfluid velocity around them is nonzero; it is quantized with the circulation quantum $\kappa_0 = 2\pi\hbar/m$. These singular lines are called *quantized vortex lines*. Note that Eq. (33) introduces a function $\Phi$ that is multivalued: It changes by $\Delta\Phi = \kappa_0 n$ after encircling each vortex line once.

For one straight vortex line along the $z$ axis of a cylindrical coordinate frame $(r, \varphi, z)$, the velocity around it is from Eq. (35) $\mathbf{v}_s = (0, v_{s\varphi}, 0)$ where

$$v_{s\varphi} = \frac{\kappa_0 n}{2\pi r} \tag{37}$$

or

$$\mathbf{v}_s = \frac{\kappa_0 n\, \mathbf{e}_z \times \mathbf{r}}{2\pi r^2} \tag{38}$$

If there are several vortex lines, the superfluid velocity is a superposition of velocities of each vortex line. For straight parallel lines,

$$\mathbf{v}_s = \sum_i \frac{\kappa_0 n_i \mathbf{e}_z \times (\mathbf{r} - \mathbf{r}_i)}{2\pi |\mathbf{r} - \mathbf{r}_i|^2} \tag{39}$$

where $\mathbf{r}_i$ is a (two-dimensional) position vector of the $i$-th vortex. Eq. (36) takes the form

$$\operatorname{curl} \mathbf{v}_s = \sum_i \kappa_0 n_i \delta^{(2)} (\mathbf{r} - \mathbf{r}_i) \tag{40}$$

Consider a large cylinder with a radius $R$. Assume that it is filled with vortex lines of the same $n$ with an uniform density $n_L$. The velocity circulation around it is from Eq. (40)

$$\oint \mathbf{v}_s\, d\mathbf{r} = \int \sum_i \kappa \delta^{(2)} (\mathbf{r} - \mathbf{r}_i)\ dS = n_L \pi R^2 \kappa$$

where $\kappa = \kappa_0 n$. This gives

$$2\pi R v_{s\varphi} = n_L \pi R^2 \kappa$$

or

$$v_{s\varphi} = n_L \kappa R/2 = \Omega R$$

The last equality holds if

$$n_L = 2\Omega/\kappa \tag{41}$$

We see that the superfluid component has on average the same velocity field as the rotating normal component provided the fluid is filled by quantized vortex lines with the density satisfying Eq. (41). Therefore, the entire liquid rotates on average with the angular velocity $\Omega$. The superfluid velocity has, of course, a highly non-uniform local distribution such that the velocity is large near each vortex line while decreasing away from it.

The energy of a straight vortex line can be calculated from Eq. (37)

$$E_L = \int \frac{\rho_s v_s^2}{2}\, dV = \frac{\kappa^2 \rho_s L}{4\pi} \int_{r_{min}}^{r_{max}} \frac{dr}{r}$$

where $L$ is the length of the line. The upper limit $r_{max}$ is determined by the larger distance at which the slow $r^{-1}$ dependence of the velocity field changes to a faster decay. In an array of vortex lines, this happens at a distance of the order of the distance between vortices, $r_0$. The intervortex distance is determined by Eq. (41)

$$r_0^2 \approx \frac{\kappa}{2\pi\Omega} \tag{42}$$

where we put $n_L = 1/\pi r_0^2$.

The lower limit is determined by the microscopic thickness of the vortex line. On the atomic scale, the vortex line has a structure characterized by the so called vortex core with the radius $r_c$. Finally, the energy of a vortex line per unit length, i.e., the linear tension, is

$$\epsilon_L = \frac{\kappa^2 \rho_s}{4\pi} \ln\left(\frac{r_0}{r_c}\right) \tag{43}$$

The vortex energy is proportional to square of circulation. Therefore, the energy of $N$ singly quantized vortices with $\kappa = \kappa_0$ is $N\epsilon_1$ where $\epsilon_1$ is the energy Eq. (43) for $n = 1$. However, the energy of one vortex with $N$ quanta, i.e., $\kappa = N\kappa_0$ is $N^2\epsilon_1$. At the same time, to create a velocity field corresponding to a given average rotation velocity $\Omega$ one needs either certain number $N$ of singly quantized vortices or one vortex with $N$ quanta. Therefore, it is more energetically favorable to have singly quantized vortices with $\kappa = \kappa_0$.

### E. Vortex near a wall. Feynman critical velocity

Consider a vortex near a flat wall of a container. Due to boundary conditions of vanishing of the normal component of flow at the wall, the vortex ends should be perpendicular to the wall. Energy minimum suggests that the vortex should have the form of a half-loop with its ends terminating at the wall (see Fig. 3).

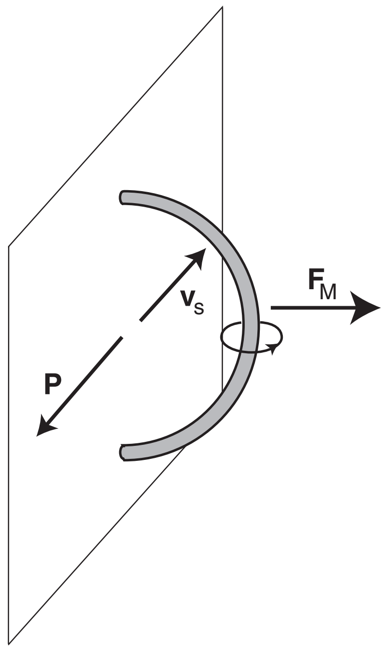


FIG. 3. The vortex half-loop attached to the wall.

The linear tension tries to contract this loop. However, if there is a flow parallel to the wall, it will try to inflate the loop. Indeed, the energy of the half-loop is from Eq. (43)

$$E = \frac{\kappa^2 \rho_s R}{4} \ln \frac{R}{r_c}$$

The momentum of the loop is (see Problem 4 to Sect. I)

$$\mathbf{P} = \kappa \rho_s \mathbf{n} S \tag{44}$$

where $S$ is the area of the loop, and $\mathbf{n}$ is the unit vector perpendicular to the plane of the loop; its positive direction is determined by the right screw rule with the vortex circulation. Assume that $\mathbf{n}$ is anti-parallel to the external superflow $\mathbf{v_s}$. For a half-circle, $S = \pi R^2/2$. The energy of the loop in the flow is then

$$E + \mathbf{P} \cdot \mathbf{v}_s = \frac{\kappa^2 \rho_s R}{4} \ln \frac{R}{r_c} - \frac{\kappa \rho_s \pi R^2}{2}$$

The energy is plotted in Fig. 4. As a function of the loop radius, the energy has a maximum at

$$R_c = \frac{\kappa}{4\pi v_s} \ln \frac{R}{r_c} \tag{45}$$

This maximum is a potential barrier

$$E_0 = \frac{\kappa^3 \rho_s}{16\pi v_s} \ln^2 \left( \frac{\kappa}{4\pi v_s r_c} \right) \tag{46}$$

that separates the state where the loop vanishes at the wall $R = 0$, $E = 0$ and the state where the loop is infinitely large (is far from the wall) $R = \infty$, $E = -\infty$. A loop with a radius smaller than $R_c$ in a flow $v_s$ will shrink while that with $R > R_c$ will grow.

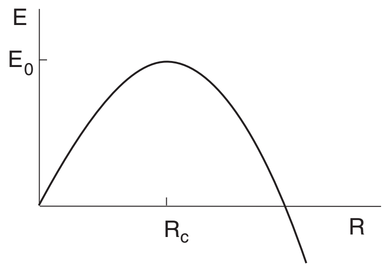


FIG. 4. The energy of the half-loop as a function of its radius.

The velocity that corresponds to an unstable equilibrium at the barrier is

$$v_c = \frac{\kappa}{4\pi R} \ln \left( \frac{R}{r_c} \right) \tag{47}$$

This is the same as the result of Problem 5 to Sect. I which also determines the equilibrium velocity of the loop. If, instead of a flat wall, we have a flow channel of a dimension $R$ the flow with a velocity from Eq. (47) will inflate loops with the maximum possible radius which is of the order of $R$. These loops will be torn away from the channel walls and will come into motion in the fluid. The vortex motion, in fact, gives rise to a dissipation due to interaction of vortices with the normal component which is called the mutual friction. As a result, there will be no superfluidity! This is why the velocity Eq. (47) is called the critical velocity of superfluidity. It was first introduced by Feynman. The exact magnitude of the critical velocity depends on the flow geometry, this is why Eq. (47) gives only an -order-of-magnitude estimate. The Feynman critical velocity is much smaller that the Landau critical velocity for creation of rotons (or phonons); it decreases for wider channels.

## Problems

### Problem 1

Find the fountain pressure head in two vessels connected by a superleak if one is heated with respect to another (see Fig 2).

### Problem 2

Find the velocity of the fourth sound, when the normal component is clamped in a narrow tube due to a large viscosity.

### Problem 3

Find the meniscus of a rotating superfluid in a vortex-free state.

### Problem 4

Find the momentum of a vortex loop with a radius $R$.

### Problem 5

Find the energy of the vortex loop and its velocity in the fluid.

### Problem 6

Find the rotation velocity above which the first vortex becomes energetically favorable in a container with a radius $R$.

## II. THE GROSS–PITAEVSKII MODEL

### A. GP equation and the coherence length

The simplest theory pretending to describe kinetic processes in superfluids is known to be the Gross–Pitaevskii theory designed for a weakly non-ideal Bose gas (with repulsion between particles) at zero temperature. It is assumed that almost all particles are in the condensed state, i.e., the wave function is $\Psi = \psi + \psi'$ where the non-condensate part of the wave function $\psi'$ is small compared to the condensate wave function $\psi$. It thus satisfies the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2 + \mu\right)\psi + \psi\int |\psi(\mathbf{r}')|^2\, U(\mathbf{r}-\mathbf{r}')\, d^3r'.$$

Here $\mu$ is the chemical potential and $U > 0$ is the interaction (repulsive) interaction. Assuming that $\psi$ varies slowly at distances of an atomic scale, one can take it out from the integral denoting

$$U_0 \equiv \int U(\mathbf{r})\, d^3r$$

The potential $U_0$ determines the scattering length $a = mU_0/4\pi\hbar^2$. For the Bose condensate of an ideal gas, $\mu = 0$. In an interacting gas, the chemical potential is known to be[2]

$$\mu = NU_0$$

where $N$ is the density of number of atoms in a non-perturbed liquid. We shall see that this expression gives the correct normalization of the condensate wave function. The Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - U_0\left(N\psi - |\psi|^2\psi\right). \tag{48}$$

The r.h.s. of this equation is nothing but the energy of the system.

This equation was obtained by Pitaevskii and Gross in 1961. The macroscopic wave function of the condensate atoms is generally complex $\psi = |\psi|e^{i\chi}$. In a spatially uniform situation $|\psi| = \sqrt{N}$ while the energy is independent of the phase $\chi$.

The particle flow is determined by the usual quantum-mechanical expression for the fluid momentum per unit volume

$$\mathbf{j} = -\frac{i\hbar}{2}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] \tag{49}$$

It appears when the phase $\chi$ varies in space. Equation (49) suggests that $\mathbf{P} = \hbar\nabla\chi$ is the momentum of a condensate particle, while

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\chi \tag{50}$$

is its velocity. The superflow is thus potential; Eq. (33) yields

$$\Phi = \frac{\hbar\chi}{m}$$

As a result, the particle current becomes $\mathbf{j} = \rho_s \mathbf{v}_s$ where $\rho_s = Nm = m|\psi|^2$ the mass density of condensate atoms. For a weakly interacting gas, $\rho_s$ coincides with $\rho$.

We can write Eq. (50) in the form of Eq. (7) used in the two-fluid description:

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla\left(\mu_s + \frac{v_s^2}{2}\right) = 0 \tag{51}$$

where

$$\mu_s = -\frac{\hbar}{m}\frac{\partial\chi}{\partial t} - \frac{v_s^2}{2} \tag{52}$$

is the chemical potential of the superfluid component. The GP equation (48) provides an expression for $\mu_s$. Let us put $\psi = |\psi|e^{i\chi}$. The complex equation (48) yields two real equations

$$|\psi|\frac{\hbar}{m}\frac{\partial\chi}{\partial t} = -|\psi|\frac{v_s^2}{2} + \frac{\hbar^2}{2m}\nabla^2|\psi| + |\psi|\mu\left(1 - |\psi|^2/N\right), \tag{53}$$

$$\frac{\partial|\psi|}{\partial t} = -\frac{\hbar}{2m}\left[\nabla|\psi|\nabla\chi + \nabla(|\psi|\nabla\chi)\right] \tag{54}$$

Equation (53) gives the expression for the chemical potential

$$\mu_s = \mu\left(|\psi|^2/N - 1\right) - \frac{\hbar^2}{2m}|\psi|^{-1}\nabla^2|\psi| \tag{55}$$

The chemical potential is zero in a spatially uniform state with $|\psi|^2 = N$. Equation (54) is nothing but the continuity equation. To see this we multiply Eq. (54) by $2m|\psi|$ and find

$$\frac{\partial(mN)}{\partial t} + \mathrm{div}\,\mathbf{j} = 0 \tag{56}$$

According to Eq. (48) the characteristic length of spatial variations of the wave function is

$$\xi = \frac{\hbar}{\sqrt{2mU_0 N}}$$

Using the expression for $a$, we find

$$\xi \sim N^{-1/3}\eta^{-1/2} \gg N^{-1/3}$$

where $\eta = aN^{1/3} \sim a/d$ is the "gas" parameter and $d$ is the distance between atoms. Since it is assumed that $\eta \ll 1$ the coherence length is indeed much longer that the interatomic distance.

This example shows that it is indeed the existence of a large parameter $\xi N^{1/3} \sim \xi/d \gg 1$ that allows one to construct a tractable model of superfluidity.

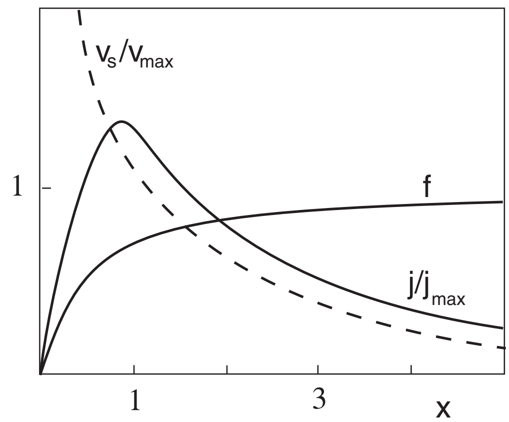


FIG. 5. The condensate density $f$, velocity $v$ and current near the vortex axis for a single-quantum vortex $n = 1$. $v_{\max}$ and $j_{\max}$ are determined by Eq. (63) below.

### B. Quantized vortex

Equation has a time-independent vortex solution. It corresponds to a wave function in the form

$$\psi = \sqrt{N}e^{in\varphi} f\left(r/\xi\right) \tag{57}$$

where $\varphi$ is the azimuthal angle in the cylindrical frame $(r, \varphi, z)$, and $n$ is an integer. With this choice of the flow potential, the function $\Phi$ becomes non-single valued: The phase $\chi = n\varphi$ varies by $2\pi n$ on going around the $z$ axis. As a result, the superfluid velocity has a nonzero circulation

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m} \oint \nabla\chi \cdot d\mathbf{r} = \frac{2\pi\hbar n}{m} \equiv n\kappa_0$$

where

$$\kappa_0 = 2\pi\hbar/m \tag{58}$$

is the quantum of circulation. It is the same as in Eq. (35) derived using semi-classical arguments in Sect I. Velocity circulation in a superfluid is quantized. The ansatz Eq. (57) corresponds to an $n$-quantum vortex.

The amplitude function $f(x)$ satisfies the equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right) - \frac{n^2 f}{x^2} + f - f^3 = 0 \tag{59}$$

For a singly quantized vortex $n = 1$, the function $f(x)$ vanishes as $f \propto x$ at the vortex axis $x \to 0$. It saturates at $f = 1$ for $x \to \infty$. The region $r \sim \xi$ near the vortex axis is called the vortex core.

The condensate velocity near the vortex is from Eq. (50)

$$\mathbf{v}_s = \frac{n\hbar\mathbf{e}_\varphi}{mr} = \frac{n\kappa_0\mathbf{e}_\varphi}{2\pi r} = \frac{n\kappa_0\mathbf{e}_z \times \mathbf{r}}{2\pi r^2}$$

where $\mathbf{e}_\varphi$ is the unit vector in the azimuthal direction while $\mathbf{e}_z$ is the unit vector in the direction of the $z$ axis. This agrees with Eq. (38). The particle current is

$$\mathbf{j} = \frac{n\hbar\rho f^2}{mr}\mathbf{e}_\varphi$$

It vanishes at $r = 0$ and decays as $1/r$ for $r \to \infty$.

More extended discussion of vortices will be given later on the basis of the Ginzburg–Landau theory.

### C. Galilean invariance, Critical velocity and Excitations

Note that Eq. (48) is Galilean invariant. If $\psi_0(\mathbf{r})$ is a solution to Eq. (48) then the function

$$\psi = \psi_0\left(\mathbf{r} - \mathbf{v}t\right) \exp\left(\frac{i}{\hbar} m\mathbf{v}\cdot\mathbf{r} - \frac{i}{\hbar}\frac{mv^2 t}{2}\right) \tag{60}$$

is also a solution to that equation carrying a particle current

$$\mathbf{j} = \mathbf{j}_0 + \rho\mathbf{v}$$

where $\mathbf{j}_0$ is the particle flow associated with the wave function $\psi_0$.

This fact has a particular meaning in terms of vortex motion. Indeed, if the system is in a state with quantized vortices in the absence of a net flow, the vortices will move together with the whole liquid with a velocity $\mathbf{v} = \mathbf{j}/\rho$ if a current $\mathbf{j}$ is driven through the system. This agrees with the Helmholtz theorem of conservation of vorticity in an ideal (non-viscous) fluid.

Taken more seriously, the Galilean invariance of the GP equation shows that one cannot define a critical velocity above which superfluidity is destroyed, in contradiction to the Landau criterion[2] of superfluidity. Consider this in more detail.

On one hand, we can look for a time independent solution

$$\psi = \sqrt{N} f e^{i\mathbf{k}\cdot\mathbf{r}} \tag{61}$$

where wave vector $k$ is related to the supercurrent through Eq. (49): $j = (\hbar\rho/m)kf^2$. Equation (48) yields

$$f^2 = 1 - k^2\xi^2 \tag{62}$$

This results in the current

$$j = (\hbar\rho/m)k(1 - k^2\xi^2)$$

This expression has a maximum at $k\xi = 1/\sqrt{3}$ with

$$j_{\max} = \frac{2\hbar\rho}{3\sqrt{3}m\xi}\ , \ v_{\max} = \frac{\hbar}{\sqrt{3}m\xi} \tag{63}$$

These quantities define the maximum supercurrent and the maximum superfluid velocity which can exist in the superfluid in a time-independent state. The time-independent state becomes unstable when the superflow exceeds $v_{\max}$.

It is interesting to compare the maximum velocity $v_{\max}$ with the sound velocity in a non-ideal Bose gas. It is known that the GP equation allows excitations which have the form of phonons at long wave lengths[2]. The excitations are perturbations of the wave function in the form of an oscillatory wave

$$\psi = \psi_0 + A\cos\left(\omega t - \mathbf{k}\cdot\mathbf{r} + \beta\right)$$

where $\psi_0$ is a stationary wave function, $A$ and $\beta$ are the amplitude and phase of oscillations. The excitation spectrum which can be found from Eq. (48) has the form (see Problem 3 to Sect. II)

$$\omega = \left[\frac{U_0 N}{m}k^2 + \left(\frac{\hbar k^2}{2m}\right)^2\right]^{1/2} \tag{64}$$

For long wave lengths or small wave vectors $k$ one obtains the sound-like dispersion

$$\omega = uk$$

with the sound velocity

$$u = \sqrt{\frac{U_0 N}{m}} = \frac{\hbar}{\sqrt{2}m\xi} \tag{65}$$

For large wave vectors $\hbar k \gg mu$, the spectrum is particle-like

$$\epsilon = \frac{p^2}{2m}$$

where $\epsilon = \hbar\omega$ and $p = \hbar k$.

The sound velocity $u$ given by Eq. (65) coincides with the Landau critical velocity defined according to the Landau criterion as

$$v_c = \min\left\{\frac{\epsilon(p)}{p}\right\} \tag{66}$$

that indeed gives the phonon velocity $u$ for the excitation spectrum $\epsilon(p) = \hbar\omega$ with $\omega$ from Eq. (64).

The Landau criterion Eq. (66) has a simple interpretation on a plot of $\epsilon$ as a function of $p$. Indeed, calculating the minimum of $\epsilon/p$ we find that it is a solution of

$$\frac{d\epsilon}{dp} = \frac{\epsilon}{p}$$

On the plot, it means that the line $\epsilon = Cp$ where $C$ is a constant is tangent to the curve $\epsilon(p)$. The constant is then $C = v_c$. The spectrum of excitations in the GP model is shown in Fig. 6 (a). The tangent coincides with the initial part of the spectrum and defines the critical velocity equal to the sound velocity. The spectrum of excitations in a real $^4$He is shown in Fig. 6 (b). It has a minimum which is historically called the "roton minimum". The critical velocity is smaller than the sound velocity.

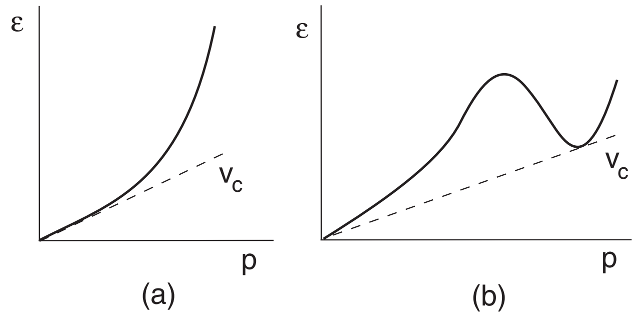


FIG. 6. The excitation spectrum in the GP model (a) and in the real $^4$He (b). The Landau critical velocity is given by the slope of the corresponding dashed line.

Equation (66) defines the velocity limit above which the excitations (phonons in our case) are produced by the moving superfluid. We can see from Eq. (65) that $v_{\max}$ is of the same order as the Landau critical velocity but it is by a factor of $\sqrt{3/2}$ larger than $v_c$:

$$v_{\max} = \sqrt{\frac{3}{2}}v_c = \sqrt{\frac{3}{2}}u. \tag{67}$$

The question is what velocity limit should be used? Is it $v_c$ or $v_{\max}$ that sets the limit of existence of a time-independent state? Or, maybe, the GP model itself breaks down at $v_c$

since creation of excitations violates the basic assumption of vanishing of non-condensate part of the wave function?

This consideration, however, contradicts to the general Galilean invariance of the GP equation. Doing the calculations of the maximum velocity, we restricted ourselves by time-independent solutions. However, the Galilean invariance tells us that, in the presence of a flow, a solution of the form of Eq. (60) also exists. If the initial state is uniform in space, the state of the system in the presence of a flow has the same (velocity-independent) amplitude and an additional phase factor

$$\exp\left(\frac{i}{\hbar}m\mathbf{v}\cdot\mathbf{r}-\frac{i}{\hbar}\frac{mv^2t}{2}\right)$$

This holds for *any* flow no matter how large its velocity is.

The apparent contradiction cannot be resolved within Eq. (48). The problem is that Eq. (48) does not include any interaction with the environment. In particular, the walls of the container are not involved. In real physical situation at a finite temperature, the environment plays a very important role since it provides a source of excitations and couples strongly to the non-condensate part of the fluid. Taken this into account we realize, that at finite temperatures, as long as the container itself does not take part in the motion, the Galilean invariance is not applicable to the entire system which also comprises effects of container walls. A zero-temperature limit has to be taken with caution, keeping in mind the relative importance of the environment and internal processes in the fluid. The issue of the proper choice of the reference frame is here of a crucial significance. Taking the laboratory frame where the container is at rest, we assume that creation of excitations dominates over the intrinsic processes far from container walls. This situation is relevant to the phenomenon of the critical velocity. One can expect the following behavior. Below the Landau critical velocity $v_c$ there are no excitations created in the fluid. The condensate part is Galilean invariant, and the state Eq. (60) is realized. Note that it is due to the fact that the *entire* system is *not* Galilean invariant that the critical limit $v_c$ does exist. Above $v_c$ excitations are created, and the validity of the GP model has to be investigated separately for each particular problem. The opposite limit of a fully Galilean-invariant behavior of the entire system assumes that effects of the environment on the bulk properties are small. Both of these pictures are only valid during some transient period whose duration is determined by properties of the fluid and its interaction with the environment. In this sense, the GP equation itself does not provide a comprehensive description of superfluids.

The main limitation of the GP equation is the lack of an explicit mechanism that establishes equilibrium towards a particular state. In principle, a source of dissipation can be identified within the GP model: it is associated with an emission of phonons. These phonons either escape to infinity or are absorbed at the walls. The efective relaxation mechanism could then be included into the GP equation in the form of a complex-valued factor $\zeta = \zeta' + i\zeta''$ instead of $\zeta = 1$ in front of the time derivative:

$$i\hbar\zeta\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - U_0\left(N\psi - |\psi|^2\psi\right). \tag{68}$$

A purely imaginary factor $\zeta$ corresponds to the so called time-dependent Ginzburg–Landau model which reasonably well describes non-stationary behavior of superconductors in some simple situations. However, the problem here is that the rate of phonon creation depends on the particular state such that it seems unlikely to construct a GP-like equation that includes an universal effective relaxation parameter $\zeta$.

## Problems

### Problem 1

Find the behavior of the wave function magnitude near the vortex axis for an $n$-quantum vortex.

### Problem 2

Show that Eq. (60) is indeed a solution of Eq. (48) if $\psi_0$ is.

### Problem 3

Find the spectrum of excitations described by Eq. (48).

## III. GINZBURG–LANDAU THEORY

Superconductivity manifests itself mainly as an absence of resistivity below some critical temperature. It is easy to measure resistivity, much easier than to measure quantities relevant to superfluid helium. The resistivity behavior as a function of temperature is shown in Fig. 7.

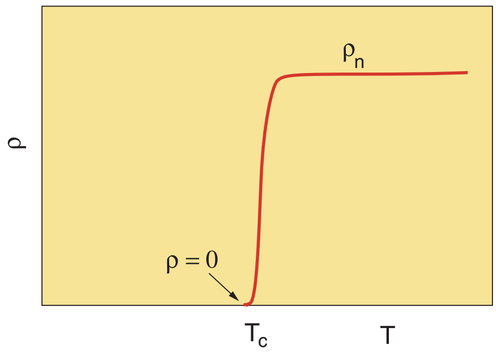


FIG. 7. Below the transition temperature, the resistivity drops to zero.

The Ginzburg-Landau theory of superconductivity created by Ginzburg and Landau in 1950[9] is based the Landau theory of second-order phase transitions[2]. The basic notion is the order parameter which describes a new property of the system that first leads to breaking of certain symmetry and then continuously develops under changing of some external parameter, for example, of the system temperature (or magnetic field, etc.). The well known example is the spontaneous magnetization of a ferromagnet. For superconductors (or superfluids) one can suggest the density of "superconducting electrons" as an order parameter. It appeared more productive, however, to introduce the probability amplitude or the"wave function" of superconducting electrons to play the role of the order parameter.

In these lecture notes we only consider $s$-wave superconductors in isotropic media. If $\Psi$ is the wave function of superconducting electrons the free energy of the system near the transition into the superconducting state can be written as an expansion in terms of a small number of superconducting electrons. If the state is spatially uniform and the magnetic field is absent, the superconducting free energy measured from the normal state has the form

$$\mathcal{F}_{sn} = V\left[a\left|\Psi\right|^2 + \frac{b}{2}\left|\Psi\right|^4\right]. \tag{69}$$

where $a$ and $b$ are the expansion parameters and $V$ is the volume of the system. At temperatures above the phase transition temperature $T_c$, the parameter $a$ should be positive. This ensures $\Psi = 0$ to be a local minimum of the free energy. Below the transition temperature, however, the coefficient $A$ should become negative. This leads to shifting the minimum to a nonzero $\Psi$. The free energy will have a minimum if the coefficient $b$ is positive:

$$\Psi_{\mathrm{min}} = |a|/b \;\; \mathcal{F}_{sn\,\mathrm{min}} = -V|a|^2/2b$$

Near the transition temperature, one can thus put the coefficient $a = a'(T - T_c)$ while $b = const.$

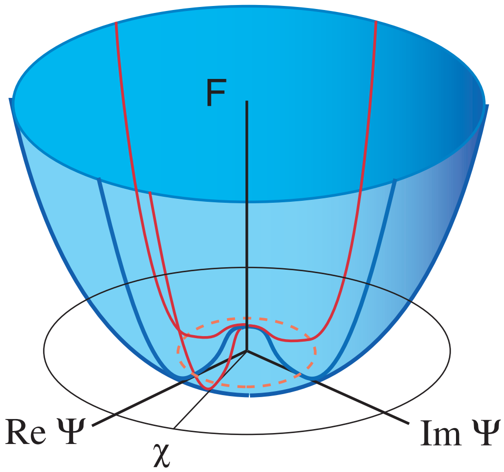


FIG. 8. Below the transition temperature, the free energy Eq. (69) has a minimum at a nonzero order parameter magnitude. The minimum energy is degenerate with respect to the order parameter phase $\chi$.

The development of the microscopic theory of superconductivity[14] has further demonstrated that the order parameter is actually a wave function of "superconducting pairs" of electrons which make a "Bose condensate". The most convenient normalization of the wave function $\Delta$ is such that its modulus is related to a certain characteristics of the quasiparticle energy spectrum in the superconductor. More specifically its modulus is chosen to be the energy gap in the electronic spectrum which opens after transition into superconducting state

$$E_p = \sqrt{(\epsilon_p - E_F)^2 + |\Delta|^2}$$

where $\epsilon_p$ is the spectrum in the normal state. The order parameter $\Delta$ is a complex function $\Delta = |\Delta| e^{i\chi}$ where $\chi$ is the same for all condensate particles if there is no current. In the presence of current and magnetic field, both the magnitude and the phase vary in space. The superconducting free energy is expanded now in terms of $\Delta$ and its gradients:

$$\mathcal{F}_{sn} = \int \left[ \alpha |\Delta|^2 + \frac{\beta}{2} |\Delta|^4 + \gamma \left| \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta \right|^2 \right] dV \tag{70}$$

Here $c$ is speed of light. We use the gaussian units for electromagnetic quantities. These units are commonly used in the physics of superconductivity. We will give the conversion to SI units in some cases. For example, in the SI units one has to put $c = 1$ in Eq. (70).

The coefficient $\gamma$ should be positive to ensure a minimum energy for a spatially homogeneous state. The total energy consists of the normal state energy $\mathcal{F}_n$, the superconducting free energy $\mathcal{F}_{sn}$, and the magnetic energy:

$$\mathcal{F} = \mathcal{F}_n + \mathcal{F}_{sn} + \int \frac{h^2}{8\pi} dV \tag{71}$$

Here $\mathbf{h}$ is the microscopic magnetic field. The average of $\mathbf{h}$ gives the magnetic induction $\mathbf{B}$. In the SI units, the magnetic energy is

$$\int \frac{\mu_0 h^2}{2} dV$$

where $\mu_0$ is permeability of vacuum. We omit the (constant) free energy of the normal state $\mathcal{F}_n$ in what follows.

The gradient term in Eq. (70) is the momentum operator in presence of the magnetic field

$$\hat{\mathbf{P}} = -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \tag{72}$$

Here the charge $2e$ accounts for the charge of a Cooper pair ($e$ is the electronic charge). It implies that there is a gauge invariance: the free energy of the system and the magnetic field do not change if one makes a simultaneous transformation

$$\chi \to \chi + f(\mathbf{r})\,,\ \ \mathbf{A} \to \mathbf{A} + \frac{\hbar c}{2e}\nabla f$$

The free energy expression is supplemented with the Maxwell equation

$$\mathrm{curl}\,\mathbf{h} = \frac{4\pi}{c}\mathbf{j} \tag{73}$$

where $\mathbf{j}$ is now the electric current and the microscopic field is

$$\mathbf{h} = \mathrm{curl}\,\mathbf{A} \tag{74}$$

In the SI units we have instead of Eq. (73)

$$\mathrm{curl}\,\mathbf{h} = \mathbf{j}.$$

At the transition temperature, $T = T_c$, the coefficient $\alpha$ changes its sign and becomes negative for $T < T_c$, while $\beta$ and $\gamma$ are positive constants. Microscopic theory gives[15,16]

$$\alpha = -\nu\,\frac{T_c - T}{T_c}\ ,\ \beta = \frac{7\zeta(3)\nu}{8\pi^2 T_c^2} \tag{75}$$

where $\nu$ is the single-spin density of states at the Fermi level, $\zeta(3) \approx 1.202$, and we use the units with $k_B = 1$. Equation (75) demonstrates that the expansion in Eq. (70) goes in the parameter $\Delta/T_c$. One has thus to assume that the order parameter is small compared to $T$.

The coefficient $\gamma$ depends on purity of the sample. The purity is characterized by the parameter $T_c\tau/\hbar$, where $\tau$ is the electronic mean free time due to the scattering by impurities. Superconductors are called clean when this parameter is large, and they are dirty in the opposite case. One has

$$\gamma = \frac{\nu\pi D}{8\hbar T_c}\,y(\tau T_c/\hbar) \tag{76}$$

where $D = v_F^2\tau/3$ is the diffusion coefficient, and

$$y(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2[(2n+1)2\pi x + 1]} \tag{77}$$

This function is $y = 1$ for a dirty limit $\tau T_c/\hbar \ll 1$, and it is

$$y(\tau T_c/\hbar) = \frac{7\zeta(3)\hbar}{2\pi^3 \tau T_c}$$

for a clean case $T_c\tau/\hbar \gg 1$. Therefore

$$\gamma = \begin{cases} \pi\nu D/8T_c\hbar\ , & \text{dirty} \\ 7\zeta(3)\nu v_F^2/48\pi^2 T_c^2\ , & \text{clean} \end{cases} \tag{78}$$

so that $\gamma_{\text{dirty}}/\gamma_{\text{clean}} \sim (\tau T_c/\hbar) \ll 1$.

### A. Ginzburg–Landau equations

Variation of $\mathcal{F}$ with respect to $\Delta$, $\Delta^*$ and $\mathbf{A}$ gives

$$\begin{aligned} \delta\mathcal{F} = \int \Bigg\{ & \left( \left[ \alpha\Delta + \beta|\Delta|^2\Delta + \gamma \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right)^2 \Delta \right] \delta\Delta^* + \text{c.c.} \right) \\ & + \left( \frac{\text{curl curl}\mathbf{A}}{4\pi} - \frac{2e\gamma}{c} \left[ \Delta^* \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta + \text{c.c.} \right] \right) \delta\mathbf{A} \\ & + \text{div} \left( \hbar\gamma \left[ \delta\Delta^* \left( \hbar\nabla - \frac{2ie}{c}\mathbf{A} \right) \Delta + \text{c.c.} \right] + \frac{1}{4\pi}\delta\mathbf{A} \times \text{curl}\mathbf{A} \right) \Bigg\} \, dV \end{aligned}$$

The requirement of extremum of the free energy gives the GL equation

$$\alpha\Delta + \beta|\Delta|^2\Delta + \gamma \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right)^2 \Delta = 0 \tag{79}$$

together with the definition of the current

$$\mathbf{j} = 2e\gamma \left[ \Delta^* \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta + \Delta \left( i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta^* \right] \tag{80}$$

which follow from the Maxwell equation (73) and Eq. (74). In addition, the surface term requires, in particular,

$$\mathbf{n} \cdot \left( \hbar\nabla - \frac{2ie}{c}\mathbf{A} \right) \Delta = 0 \tag{81}$$

where $\mathbf{n}$ is the unit vector along the normal to the surface. This is the so called "natural" boundary condition. In particular, it tells that the current through the surface vanishes. It only applies at the boundary with vacuum. In other cases such that contacts with conductors, etc., one has to include also the energy of interaction between the superconductor and contacting media, which will change the boundary conditions. For the same reason, the vector-potential term can not be put to zero independently from the corresponding contribution of external fields.

The first equation (79) is very similar to the time-independent version of the Gross–Pitaevskii equation (48) for a noncharged system, $e = 0$. The particle current $\mathbf{j}/e$ from Eq. (80) also coincides with Eq. (49) in the absence of charge.

### B. Discussion of the GL equations

Consider first the equation for the order parameter Eq. (79). In a homogeneous case without a current and a magnetic field, it gives

$$\Delta = \Delta_{\rm GL} = \sqrt{|\alpha|/\beta} = \left(\frac{8\pi^2}{7\zeta(3)}\right)^{\frac{1}{2}} T_c \, (1 - T/T_c)^{\frac{1}{2}} \tag{82}$$

As we already know, the ratio $\Delta_{\rm GL}/T_c$ should be small. This implies that the GL theory works for temperatures close to $T_c$ such that $1 - T/T_c \ll 1$.

The free energy density in a homogeneous case is

$$F_c = -|\alpha|^2/2\beta \tag{83}$$

It is called the condensation energy. It defines the thermodynamic critical magnetic field

$$H_c^2 = 4\pi|\alpha|^2/\beta \tag{84}$$

when the magnetic energy is equal to the condensation energy, i.e., to the energy in the absence of magnetic fields in the bulk superconductor (the so called Meissner state, see a discussion later). In the SI units

$$H_c^2 = |\alpha|^2/\mu_0\beta$$

The thermodynamic critical field is linear in temperature

$$H_c(T) \sim H_c(0)\left(1 - \frac{T}{T_c}\right)$$

where $H_c(0)$ is formally defined by Eq. (84) where we put $T = 0$ in the coefficient $\alpha$.

Above the field $H_c$, the superconducting state without currents has a larger energy than the normal state. Indeed, the proper thermodynamic potential in an applied field is the Gibbs free energy

$$\mathcal{G} = \mathcal{F} - \int \frac{\mathbf{H} \cdot \mathbf{B}}{4\pi}$$

In the superconducting state $B = h = 0$, and the Gibbs free energy density is $G_s = F_{sn} = F_c = -|\alpha|^2/2\beta$. In the normal state, $F_{sn} = 0$, $B = h = H$ so that $G_n = -H^2/8\pi$. The energy in the normal state becomes smaller than $G_s$ for $H > H_c$.

Eq. (79) defines the length

$$\xi(T) = \sqrt{\gamma\hbar^2/|\alpha|} \propto (1 - T/T_c)^{-\frac{1}{2}} \tag{85}$$

which is a characteristic scale of variations of the order parameter. It is called the coherence length. In the clean case $T_c\tau/\hbar \gg 1$ the coherence length is

$$\xi(T) = \left(\frac{7\zeta(3)}{12}\right)^{\frac{1}{2}} \xi_0 \, (1 - T/T_c)^{-\frac{1}{2}} \tag{86}$$

where

$$\xi_0 = \hbar v_F/(2\pi T_c) \tag{87}$$

is the "zero-temperature" coherence length. In the dirty case

$$\xi(T) = \frac{\pi\sqrt{\xi_0 \ell}}{2\sqrt{3}} \, (1 - T/T_c)^{-\frac{1}{2}} \tag{88}$$

where $\ell = v_F \tau$ is the electron mean free path. The impurity parameter can be expressed through the ratio of $\xi_0$ and $\ell$:

$$T_c\tau/\hbar = \ell/2\pi\xi_0 \tag{89}$$

so that a dirty limit corresponds to $\ell \ll \xi_0$ while a clean limit is for $\ell \gg \xi_0$. Using $\xi(T)$ and $\Delta_0$ one can write Eq. (79) in the form

$$\xi^2\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)^2 \Delta + \Delta - \Delta|\Delta|^2/\Delta_{\rm GL}^2 = 0 \tag{90}$$

which contains only one parameter $\xi$.

Let us put $\Delta = |\Delta|e^{i\chi}$. The complex equation (90) gives two real equations. One is

$$\xi^2\left(\nabla^2|\Delta| - \frac{4e^2}{\hbar^2 c^2}\mathbf{Q}^2|\Delta|\right) + |\Delta| - |\Delta|^3/\Delta_{\rm GL}^2 = 0 \tag{91}$$

Here we introduce the gauge invariant vector potential

$$\mathbf{Q} = \mathbf{A} - \frac{\hbar c}{2e}\nabla\chi \tag{92}$$

The other equation is the conservation of supercurrent

$$\operatorname{div}\mathbf{j}_s = 0 \tag{93}$$

which agrees with the Maxwell equation (73).

Consider now the expression for current Eq.(80). Using the definition of the momentum operator Eq.(72) we introduce the superconducting velocity operator

$$2m\hat{\mathbf{v}}_s = \hat{\mathbf{P}} \tag{94}$$

for a Cooper pair with the mass $2m$. Now the current becomes

$$\mathbf{j} = -\frac{e^2 N_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) = N_s e\mathbf{v}_s \tag{95}$$

where

$$\mathbf{v}_s = \frac{\hbar}{2m}\left(\nabla\chi - \frac{2e}{\hbar c}\mathbf{A}\right) = -\frac{e}{mc}\mathbf{Q} \tag{96}$$

and

$$N_s = 8m\gamma|\Delta|^2 \tag{97}$$

is the density of "superconducting electrons". For low currents or magnetic fields, the order parameter does not depend on the current, $|\Delta| = \Delta_{\rm GL}$. In the clean case

$$N_s = 8m\gamma\Delta_{\rm GL}^2 = 8m\gamma|\alpha|/\beta = \frac{8\nu E_F}{3}\left(1-\frac{T}{T_c}\right) = 2N\left(1-\frac{T}{T_c}\right) \tag{98}$$

where $N = p_F^3/3\pi^2$ is the total number of electrons. Equation (98) is the same as $N_s = N\left(1-\frac{T^2}{T_c^2}\right)$. The last equality in Eq. (98) holds for a simple metal with a parabolic spectrum $\epsilon_p = p^2/2m$. For a dirty case

$$N_s = \frac{16\pi^3\nu E_F}{21\zeta(3)}\frac{T_C\tau}{\hbar}\left(1-\frac{T}{T_c}\right) = \frac{4\pi^3}{7\zeta(3)}\frac{T_c\tau}{\hbar}\left(1-\frac{T}{T_c}\right)N. \tag{99}$$

It is much smaller than in the clean case: the scattering on impurities impedes the supercurrent which effectively leads to a reduction in the superconducting density. Note that the critical temperature of a $s$-wave superconductor $T_c$ is itself insensitive to the impurities.

The Maxwell equation Eq.(73) combined with Eq. (95) gives

$$\text{curl curl}\,\mathbf{A} = -\frac{4\pi N_s e^2}{mc^2}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) \tag{100}$$

or

$$\text{curl curl}\,\mathbf{A} = -\lambda_L^{-2}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) \tag{101}$$

where we define the characteristic length

$$\lambda_L = \left(\frac{mc^2}{4\pi N_s e^2}\right)^{\frac{1}{2}} \tag{102}$$

In SI units,

$$\lambda_L = \left(\frac{m}{\mu_0 N_s e^2}\right)^{\frac{1}{2}}$$

For low currents

$$\lambda_L = \left(\frac{c^2\beta}{32\pi e^2\gamma|\alpha|}\right)^{\frac{1}{2}} \propto \left(1-\frac{T}{T_c}\right)^{-\frac{1}{2}} \tag{103}$$

The length $\lambda_L$ is called the London penetration length. It determines the characteristic scale of variations of the magnetic field. Now the current can be written as

$$\mathbf{j} = -\frac{c}{4\pi\lambda_L^2}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) \tag{104}$$

We define the Ginzburg-Landau current

$$j_{GL} = 4e\hbar\gamma\Delta_{\mathrm{GL}}^2/\xi = c^2/(8\pi e\lambda^2\xi) \tag{105}$$

It sets the order of magnitude of the largest current which the superconductor can sustain. The maximum current is called also the *pair-breaking current* since superconductivity is destroyed by larger currents. The pair-breaking current corresponds to the critical current, Eq. (63), with the largest possible gradient of the order parameter phase $\nabla\chi \sim 1/\xi$.

With the Ginzburg–Landau equation (79) we can transform the free energy expression Eq. (70) to another form. Let us perform integration by parts in the gradient term and substitute the kinetic energy term using Eq. (79). We obtain

$$\mathcal{F} = \int \left[-\frac{\beta}{2}|\Delta|^4 + \frac{h^2}{8\pi}\right] dV + \hbar^2\gamma \int d\mathbf{S}\Delta^* \left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)\Delta$$

The surface term vanishes because of the boundary conditions Eq. (81). We get

$$\mathcal{F} = \int \left[-\frac{\beta}{2}|\Delta|^4 + \frac{h^2}{8\pi}\right] dV \tag{106}$$

Sometimes, it is convenient to use the normalization of the order parameter such that it has the form of the wave function $\Psi$ of superconducting electrons. The free energy becomes

$$\mathcal{F}_{sn} = \int \left[a\left|\Psi\right|^2 + \frac{b}{2}\left|\Psi\right|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Psi\right|^2\right] dV. \tag{107}$$

The constants $a$ and $b$ satisfy

$$\frac{|a|}{b} = \frac{mc^2}{16\pi e^2\lambda_L^2}$$

and determine the new order parameter magnitude $|\Psi_{\mathrm{GL}}|^2 = |a|/b$ which is

$$|\Psi_{\mathrm{GL}}|^2 = 2m\gamma\Delta_{\mathrm{GL}}^2$$

in terms of the previous definition of $\Delta$. The coherence length is now expressed through the electronic mass

$$\xi^2 = \frac{\hbar^2}{2m|a|}.$$

More discussion of the GL equations can be found in Ref.[4].

### C. Fluctuations

We have seen that the free energy expansion Eq. (70) holds when the parameter $\Delta/T_c$ is small, i.e., when $1 - T/T_c \ll 1$. However, being a mean-field theory, the GL theory cannot be valid very close to the transition temperature because of an increasing magnitude of fluctuations. Let us calculate the average $|\Delta|^2$ due to fluctuations at temperatures slightly below $T_c$. The variation of free energy in a volume of the order of $\xi^3$ near the equilibrium value, $|\Delta| = \Delta_{\rm GL} + \delta|\Delta|$, is

$$\delta\mathcal{F} = \frac{1}{2}\left(\frac{\delta^2\mathcal{F}}{\delta|\Delta|^2}\right)(\delta|\Delta|)^2 = 4\xi^3|\alpha|\,(\delta|\Delta|)^2$$

The average can be calculated with the probability of fluctuations

$$P\,(\delta|\Delta|) = C\exp\left\{-\delta\mathcal{F}/T\right\}$$

where the normalization constant $C$ is found from

$$\int C\exp\left\{-\delta\mathcal{F}/T\right\}d\,(\delta|\Delta|) = 1 \tag{108}$$

We obtain

$$\begin{aligned}\left\langle(\delta|\Delta|)^2\right\rangle &= \int C\exp\left\{-\delta\mathcal{F}/T\right\}(\delta|\Delta|)^2\,d\,(\delta|\Delta|)\\ &= \int C\exp\left\{-\frac{4\xi^3|\alpha|\,(\delta|\Delta|)^2}{T}\right\}(\delta|\Delta|)^2\,d\,(\delta|\Delta|)\\ &= 2T/|\alpha|\xi^3\end{aligned}$$

This fluctuation should be smaller than $\Delta_{\rm GL}^2$, i.e., $2T/|\alpha|\xi^3 \ll |\alpha|/\beta$ or

$$\frac{2T}{\xi^3} \ll \frac{|\alpha|^2}{\beta}$$

This can be written as

$$\left|1 - \frac{T}{T_c}\right| \gg \left[\frac{T_c}{H_c^2(0)\xi_0^3}\right]^2 \equiv Gi \tag{109}$$

Equation (109) is the Ginzburg criterion[10,11]; the dimensionless quantity $Gi$ is called the Ginzburg number. It is the second power of the ratio of the critical temperature and the zero-temperature condensation energy in the volume of a cube with the size of the coherence length. Using the microscopic values for the GL parameters, we obtain

$$Gi \sim \frac{T_c^4}{E_F^4} \sim \frac{\hbar^4}{(\xi_0 p_F)^4} \tag{110}$$

We can see that the Ginzburg number is very small for superconductors where $\xi_0 p_F/\hbar \sim 10^2 \div 10^3$. Therefore, Eq. (109) can be easily fulfilled for superconductors: In superconductors, there exists a broad region near $T_c$ where fluctuations are not important.

We observe again that the applicability of the GL theory depends crucially on the existence of the large parameter $\xi_0 p_F/\hbar \sim \xi(T)/d$. For helium II, as we know, such parameter does not exist, thus one cannot directly apply a GL-type description[17] (see[18]) to helium II except for the GP model for weakly non-ideal Bose gas, discussed in Section II, which fortunately does have such a parameter.

### D. The Ginzburg–Landau parameter. Type I and type II superconductors

The ratio of the two characteristic lengths is called the Ginzburg-Landau parameter

$$\kappa = \frac{\lambda_L(T)}{\xi(T)} = \left[\frac{\beta c^2}{32\pi\hbar^2 e^2 \gamma^2}\right]^{\frac{1}{2}} \tag{111}$$

It is independent of $T$ and is determined by the material characteristics.

For clean superconductors it is

$$\kappa = \left(\frac{9\pi^4}{14\zeta(3)}\right)^{\frac{1}{2}} \left[\left(\frac{a_0 p_F}{2\pi\hbar}\right)\left(\frac{e^2/a_0}{E_F}\right)\right]^{\frac{1}{2}} \frac{\hbar c}{e^2}\frac{T_c}{E_F} \tag{112}$$

Here $a_0$ is the interatomic distance. Usually, it is of the order of $1/2\pi\hbar p_F$. The ratio of the Coulomb interaction energy of conducting electrons, $e^2/a_0$, to the Fermi energy is of the order of unity for good metals, but it may become larger for systems with strong correlations between the electrons. The last factor in Eq. (112) is usually small: $T_c/E_F$ is of the order of $10^{-3}$ for usual superconductors, but it is of the order of $10^{-1} \div 10^{-2}$ for high temperature superconductors with $T_c \sim 100K$ and $E_F \sim 1000K$. The fine structure constant $e^2/\hbar c = 1/137$.

We see that for usual clean superconductors the Ginzburg-Landau parameter is normally small $\kappa \ll 1$, though, in some cases it may be of the order of 1. On the contrary, for high temperature superconductors, which have a tendency to be strongly correlated systems with a not very low ratio of $T_c/E_F$, the parameter $\kappa$ is usually very large. The Ginzburg-Landau parameter increases for dirty superconductors:

$$\kappa_{\text{dirty}} \sim \kappa_{\text{clean}}(\hbar/T_c\tau) \tag{113}$$

Therefore, dirty alloys normally have a large $\kappa$.

The magnitude of $\kappa$ divides all superconductors between two types: type I and type II superconductors. Those with $\kappa < 1/\sqrt{2}$ belong to the type I, while those with $\kappa > 1/\sqrt{2}$ are type II superconductors.

### E. Meissner effect. Magnetic flux quantization

Eq. (101) describes the Meissner effect, i.e., an exponential decay of weak magnetic fields and supercurrents in a superconductor. The characteristic length over which the magnetic field decreases is just $\lambda_L$. Consider a superconductor which occupies the half-space $x > 0$. A magnetic field $h_y$ is applied parallel to its surface (Fig. 9). Taking curl of Eq. (101) we obtain

$$\frac{\partial^2 h_y}{\partial x^2} - \lambda_L^{-2} h_y = 0$$

which gives $h_y = h_y(0)\exp(-x/\lambda)$. The field decays in a superconductor such that there is no field in the bulk. The supercurrent also decays and vanishes in the bulk acording to Eq. (73).

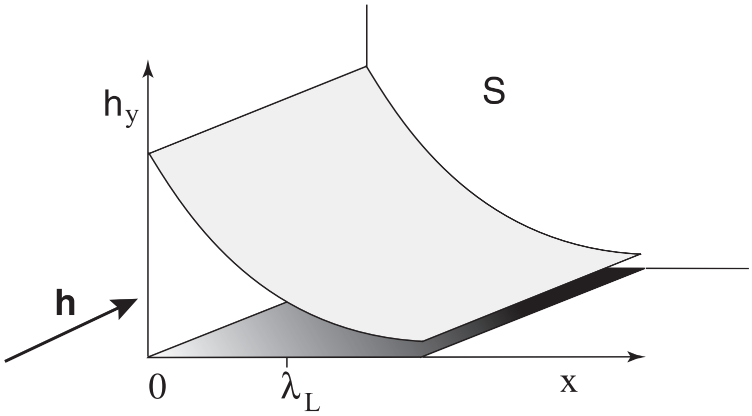


FIG. 9. The Meissner effect: Magnetic field penetrates into a superconductor only over distances shorter than $\lambda_L$.

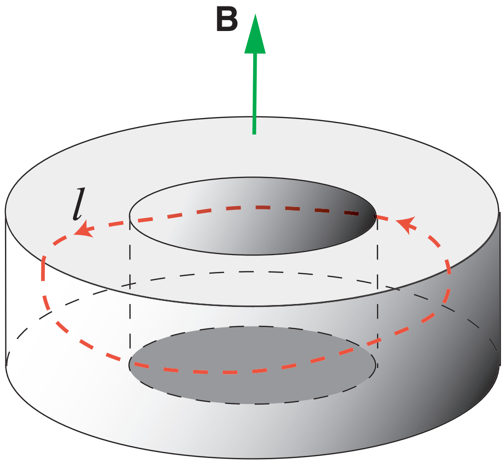


FIG. 10. Magnetic flux through the hole in a superconductor is quantized.

Therefore,

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = 0$$

in a bulk superconductor. The magnetization and susceptibility are

$$\mathbf{M} = -\mathbf{H}/4\pi\,;\ \chi = \frac{\partial M}{\partial H} = -\frac{1}{4\pi} \tag{114}$$

as for an ideal diamagnetic. The Meissner effect in type I superconductors persists up to the field $H = H_{c1}$ above which superconductivity is destroyed, see Fig. 19. Type II superconductors display the Meissner effect up to much lower fields, after which vortices appear (see the following section).

Let us consider an non-singly-connected superconductor with dimensions larger than $\lambda_L$ placed in a magnetic field (Fig. 10). We choose a contour which goes all the way inside the superconductor around the hole and calculate the contour integral

$$\oint \left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) d\mathbf{l} = \int_S \text{curl}\,\mathbf{A}\,dS - \frac{\hbar c}{2e}\Delta\chi = \Phi - \frac{\hbar c}{2e}2\pi n \tag{115}$$

Here $\Phi$ is the magnetic flux through the contour. The phase change along the closed contour is $\Delta\chi = 2\pi n$ where $n$ is an integer because the order parameter is a single valued function. Since $\mathbf{j} = 0$ in the bulk, we obtain $\Phi = \Phi_0 n$ where

$$\Phi_0 = \frac{\pi\hbar c}{e} \approx 2.07 \cdot 10^{-7}\ \text{Oe} \cdot \text{cm}^2 \tag{116}$$

is the quantum of magnetic flux. In SI units, $\Phi_0 = \pi\hbar/e$.

## Problems

### Problem 1

Estimate, in terms of the microscopic parameters, the hight of the energy barrier one needs to overcome to inflate a vortex loop, Eq. (46), in a superconductor for a maximum possible superfluid velocity $v_s \sim \hbar/m\xi$. Compare it with the temperature $T_c$. What is the probability of such a fluctuation?

### Problem 2

Find the jump of the specific heat at the superconducting transition.

### Problem 3

Find the behavior of the order parameter near a contact with the normal region. Magnetic field and currents are absent.

### Problem 4

Find the surface energy of the boundary with the normal region. Magnetic field and currents are absent.

## IV. VORTICES IN TYPE II SUPERCONDUCTORS

Vortices are the objects which play a very special role in superconductors and superfluids. In superconductors, each vortex carries exactly one magnetic-flux quantum. Being magnetically active, vortices determine the magnetic properties of superconductors. In addition, they are mobile if the material is homogeneous and there are no defects which can attract vortices and "pin" them somewhere in the superconductor. In fact, a superconductor in the vortex state is no longer superconducting in a usual sense. Indeed, there is no complete Meissner effect: some magnetic field penetrates into the superconductor via vortices. In addition, regions with the normal phase appear: since the order parameter turns to zero at the vortex axis and is suppressed around each vortex axis within a vortex core with a radius of the order of the coherence length, there are regions with a finite low-energy density of states. Moreover, mobile vortices come into motion in the presence of an average (transport) current. We shall see that there is a finite resistivity (the so-called flux flow resistivity): a superconductor is no longer "superconducting"! This is certainly an important effect.

In superfluids, vortices appear in a container with helium rotating at an angular velocity $\Omega$ above a critical value which is practically not high and can easily be reached in experiment[7]. Vortices are also created if a superfluid flows in a tube with a sufficiently high velocity. The driving force that pushes vortices is now the Magnus force. Vortices move and experience reaction from the normal component; this couples the superfluid and normal components and produces a "mutual friction" between them. As a result, the superflow is no longer persistent. Remember the definition of superfluidity: a liquid is superfluid if it can flow without friction through a narrow tube. One needs to restrict the fluid to a narrow tube because it is a narrow tube that inhibits vortex motion thus switching off the dissipation. As we can see, vortex dynamics is the very heart of superfluidity. This was realized by Feynman in the early days of superfluidity[19].

### A. Transition into superconducting state in a magnetic field

Vortices play an important role also in bringing about the superfluid phase transition itself. Let us put a sample of a type-II superconductor at a temperature below $T_c$ into a high magnetic field and start to decrease the applied field $H$. At some field magnitude, the sample will become superconducting. We shall see that this transition is of the second order. Close to the transition point thus $\Delta \ll \Delta_{\rm GL}$. Let the magnetic field be along the $z$ axis (see Fig. 11).

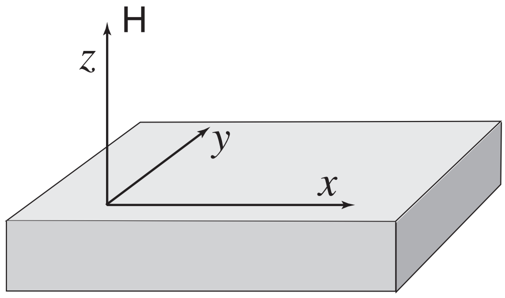


FIG. 11.

We can linearize the GL equations in a small $\Delta$:

$$\xi^2 \left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)^2 \Delta + \Delta = 0 \tag{117}$$

The vector potential can be taken in the Landau gauge $\mathbf{A} = (0,\, Hx,\, 0)$. The order parameter depends on $x$ and $y$. Now we have

$$\frac{\partial^2 \Delta}{\partial x^2} + \left(\frac{\partial}{\partial y} - \frac{2ieHx}{\hbar c}\right)^2 \Delta + \xi^{-2}\Delta = 0 \tag{118}$$

This is the well known Schrödinger equation for a charge in a magnetic field. We put

$$\Delta = e^{iky} f(x)$$

and obtain the oscillator equation

$$\frac{\partial^2 f}{\partial x^2} - \left(k - \frac{2eHx}{\hbar c}\right)^2 f + \xi^{-2} f = 0 \tag{119}$$

with

$$\omega_0 = \frac{2eH}{mc}\,,\ E = \frac{\hbar^2}{2m\xi^2}$$

The energy spectrum $E = \hbar\omega_0(n + 1/2)$ is

$$\frac{\hbar}{2m\xi^2} = \frac{2eH}{mc}\left(n + \frac{1}{2}\right)$$

The highest $H = H_{c2}$ is for $n = 0$:

$$H_{c2} = \frac{\hbar c}{2e\xi^2} = \frac{\Phi_0}{2\pi\xi^2} \propto 1 - \frac{T}{T_c}. \tag{120}$$

In SI units,

$$H_{c2} = \frac{\Phi_0}{2\pi\mu_0\xi^2}$$

It is the upper critical magnetic field below which the transition into superconducting state occurs. Comparing it with $H_c$ we observe that

$$H_{c2} = \sqrt{2}\kappa H_c = \sqrt{8\pi\beta}\kappa\Delta^2_{\mathrm{GL}}$$

For type II superconductors with $\kappa > 1/\sqrt{2}$, the upper critical field $H_{c2} > H_c$: transition occurs to the state which, as we will see soon, has persistent currents in the superconducting bulk.

The solution for the lowest energy level is a Gaussian function

$$f = C \exp\left[-\frac{1}{2\xi^2}\left(x - \frac{\hbar c k}{2eH_{c2}}\right)^2\right] \tag{121}$$

The solution Eq. (121) is centered at $x = \hbar ck/2eH$. Actually, the full solution is a linear combination of these solutions for different $k$. One can construct a periodic solution in a form

$$\Delta = \sum_n C_n e^{iqny} \exp\left[-\frac{1}{2\xi^2}\left(x - \frac{\hbar cqn}{2eH_{c2}}\right)^2\right] \tag{122}$$

It is periodic in $y$ with the period $2\pi/q$. It would be periodic in $x$ as well if the coefficients $C_n$ satisfy periodicity condition $C_{n+p} = C_n$. Then,

$$\Delta\left(x+\frac{p\hbar cq}{2eH_{c2}},\, y\right) = e^{ipqy}\Delta\left(x,\, y\right)$$

One sees that $|\Delta|$ is periodic with the period

$$X_0 = \frac{\hbar cpq}{2eH_{c2}}$$

The simplest case is realized when all the coefficients $C$ are equal, $p = 1$. The array forms a rectangular lattice. The periods are

$$X_0 = \frac{\hbar cq}{2eH_{c2}},\ Y_0 = \frac{2\pi}{q}.$$

The unit cell area is

$$X_0Y_0 = \Phi_0/H_{c2} = 2\pi\xi^2,$$

which corresponds to exactly one flux quantum per unit cell. If $q$ is chosen in such a way that $X_0 = Y_0$, we obtain a square lattice.

However, the period $2\pi/q$ is not the least possible period in $y$ for $p \neq 1$. The analysis shows that if $p = 2$ and

$$C_0 = \pm iC_1$$

the period in $y$ is

$$Y_0 = \frac{\pi}{q}$$

The unit cell area is

$$X_0Y_0 = \Phi_0/H_{c2} = 2\pi\xi^2$$

which again corresponds to exactly one flux quantum per unit cell. If $X_0/Y_0 = 2/\sqrt{3}$ one obtains a hexagonal lattice (see Fig. 12).

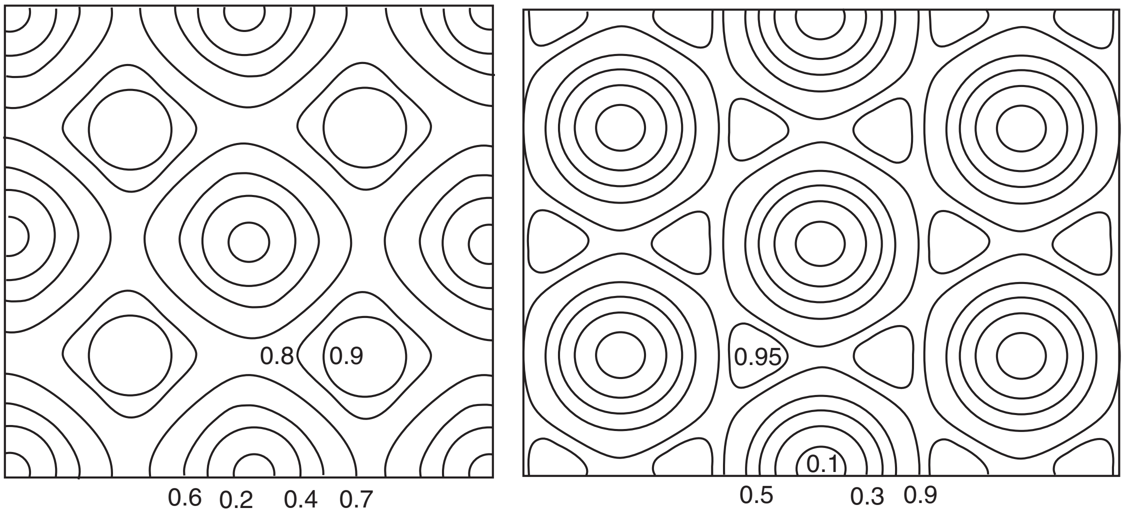


FIG. 12. Left panel: Lines of constant $|\Delta|$ in a square lattice according to Ref.[20]. Right panel: The same for a hexagonal lattice according to Ref.[21].

The $|\Delta|$-pattern has zeroes at the points $x = X_0/2 + X_0 n$, $y = Y_0/2 + Y_0 m$, surrounded by current lines. Indeed, the current is

$$
\begin{aligned}
j_x &= -\frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_{\mathrm{GL}}^2}\left[i\Delta^*\frac{\partial\Delta}{\partial x} - i\Delta\frac{\partial\Delta^*}{\partial x}\right] \\
j_y &= -\frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_{\mathrm{GL}}^2}\left[\Delta^*\left(i\frac{\partial\Delta}{\partial y} + \frac{2eH_{c2}x}{\hbar c}\Delta\right) - \Delta\left(i\frac{\partial\Delta^*}{\partial y} - \frac{2eH_{c2}x}{\hbar c}\Delta\right)\right]
\end{aligned}
$$

To transform this further we use the identity which holds for the function of the type of Eq. (122):

$$
\frac{\partial\Delta}{\partial x} = \left(-i\frac{\partial}{\partial y} - \frac{2eH_{c2}x}{\hbar c}\right)\Delta \tag{123}
$$

With help of Eq. (123) we get

$$
j_x = -\frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_{\mathrm{GL}}^2}\frac{\partial|\Delta|^2}{\partial y} \tag{124}
$$

$$
j_y = \frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_{\mathrm{GL}}^2}\frac{\partial|\Delta|^2}{\partial x} \tag{125}
$$

These expressions suggest that $|\Delta(x,y)|^2$ is a stream function, i.e., that the current flows along the lines of constant $|\Delta|$. If we place the node of $|\Delta|$ in the middle of the Bravais unit cell, then the current along the boundary of a unit cell is zero: due to periodicity, the lines of constant $|\Delta|$ are perpendicular to the boundary (see Fig. 13). We now calculate the contour integral of $\mathbf{A} - (\hbar c/2e)\nabla\chi = 0$ and obtain $\Delta\chi = 2\pi$ since the flux through the unit cell is equal to one flux quantum. The phase of the order parameter acquires the increment of $2\pi$ after encircling the point where the order parameter is zero.

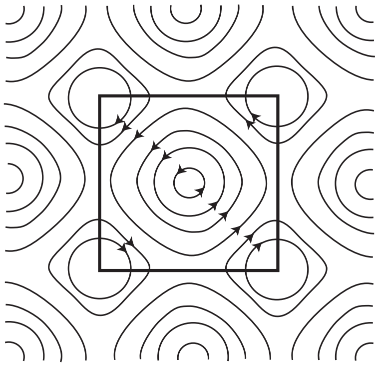

FIG. 13. The Bravais unit cell for a square lattice. Arrows show the flow pattern of the supercurrent; the current lines are perpendicular to the unit-cell boundary.

Here we come to a vortex: *A quantized vortex is a linear (in three dimensions) object which is characterized by a quantized circulation of the order parameter phase around this line.* We see that transition into a superconducting state in a magnetic field below $H_{c2}$ gives rise to formation of vortices. Vortices in superconductors were theoretically predicted by Abrikosov[20] in 1957.

Let us consider a magnetic field slightly below $H_{c2}$ such that $H_{c2}-H \ll H_{c2}$. The solution of the GL equation is $\Delta = \Delta_0 + \Delta_1$ where $\Delta_0$ is the solution Eq. (122) of the linearized equation and $\Delta_1$ is a small correction $\Delta_1$. This correction is caused by (i) nonlinear term in the GL equation, (ii) variations in $\mathbf{A}$ due to the supercurrent Eqs. (124,125), and (iii) deviation of $H$ from $H_{c2}$.

Using the Maxwell equation

$$j_x = \frac{c}{4\pi}\frac{\partial h_z}{\partial y},\ j_y = -\frac{c}{4\pi}\frac{\partial h_z}{\partial x}$$

we obtain from Eqs. (124,125)

$$\delta h_z = -\frac{\hbar c|\Delta_0|^2}{4\lambda_L^2 e\Delta_{\rm GL}^2} \tag{126}$$

Therefore, the vector potential becomes $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$ where $\mathbf{A}_0 = (0,\ H_{c2}x,\ 0)$,

$$\mathbf{A}_1 = (0,\, (H - H_{c2})\, x,\, 0) + \delta\mathbf{A}$$

and $\delta\mathbf{A}$ is due to the correction to the magnetic field induced by the supercurrent such that $\delta\mathbf{h} = \mathrm{curl}\,\delta\mathbf{A}$. As a result, $h_{1z} = H - H_{c2} + \delta h_z$ and $h_z = H + \delta h_z$.

Since the non-disturbed function of Eq.(122) satisfies the linearized GL equation with $\mathbf{A} = \mathbf{A}_0$, we obtain for the correction $\Delta_1$

$$\begin{aligned} &\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)^2 \Delta_1 + \xi^{-2}\Delta_1 \\ -\frac{2ie}{\hbar c}\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)(\mathbf{A}_1\Delta_0) - &\frac{2ie}{\hbar c}\mathbf{A}_1\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)\Delta_0 \\ &-\xi^{-2}|\Delta_0|^2\Delta_0/\Delta_{\mathrm{GL}}^2 = 0 \end{aligned}$$

Now we multiply this equation by $\Delta_0^*$ from Eq. (122) and integrate it over $dx\,dy$. After integration by parts using Eqs. (123), (126) we obtain

$$\frac{2e}{\hbar c}\int |\Delta_0|^2(H - H_{c2} + \delta h)\, dx\, dy + \Delta_{\mathrm{GL}}^{-2}\xi^{-2}\int |\Delta_0|^4\, dx\, dy = 0 \tag{127}$$

We introduce the average

$$\langle|\Delta|^2\rangle = S^{-1}\int |\Delta|^2\, dx\, dy$$

Using Eq. (126) we obtain

$$\Delta_{\mathrm{GL}}^2\langle|\Delta_0|^2\rangle\left(1 - \frac{H}{H_{c2}}\right) = \left(1 - \frac{1}{2\kappa^2}\right)\langle|\Delta_0|^4\rangle \tag{128}$$

It is convenient to introduce the ratio

$$\beta_A = \frac{\langle|\Delta|^4\rangle}{\langle|\Delta|^2\rangle^2} \geq 1 \tag{129}$$

The value $\beta_A$ is determined by the structure of the vortex lattice. Now we get

$$\langle|\Delta_0|^2\rangle = \frac{2\kappa^2\Delta_{\mathrm{GL}}^2}{(2\kappa^2 - 1)\beta_A}\left(1 - \frac{H}{H_{c2}}\right) \tag{130}$$

Eq. (130) shows that $|\Delta|^2$ has a small magnitude proportional to $1 - H/H_{c2}$ if $\kappa > 1/\sqrt{2}$. For $\kappa < 1/\sqrt{2}$ the order parameter $\Delta$ for $H$ close to $H_{c2}$ is no longer small: it jumps to a finite value making the transition a first-order transition.

*Type II superconductors have $\kappa > 1/\sqrt{2}$ and experience a second-order transition into superconducting state below the upper critical field with formation of quantized vortices.*

Using Eq. (126) we can calculate the magnetization of the superconductor. The magnetic induction is $B = \langle h_z \rangle = H + \langle \delta h_z \rangle$. Then

$$M_z = \frac{B - H}{4\pi} = \frac{\langle \delta h_z \rangle}{4\pi} = -\frac{H_{c2} - H}{4\pi \beta_A (2\kappa^2 - 1)} \tag{131}$$

The applied field can be expressed through the magnetic induction

$$H = H_{c2} - \frac{\beta_A (2\kappa^2 - 1)(H_{c2} - B)}{1 + \beta_A (2\kappa^2 - 1)}$$

Using Eqs. (129), (130), and (126) one can calculate the free energy density Eq. (106). As a function of the magnetic induction it becomes

$$F = \frac{B^2}{8\pi} - \frac{(H_{c2} - B)^2}{8\pi [1 + (2\kappa^2 - 1)\beta_A]} \tag{132}$$

For a given $B$, the free energy decreases with decreasing $\beta_A$. For a square lattice $\beta_A = 1.18$ while for a hexagonal lattice $\beta_A = 1.16$. Actually, a square lattice is an unstable configuration: it corresponds to an extremum rather than to a minimum of the free energy. The stable configuration is a hexagonal lattice (Fig. 12).

### B. Single vortex

The previous case corresponds to the situation where vortices are closely packed together: the distance between them is of the order of the coherence length. Let us now consider an example of an isolated single-quantum vortex when the distance to the neighbor vortex is much larger than $\xi$. Since each vortex unit cell with an area $S$ carries one magnetic flux quantum, $\Phi_0 = SB$ the intervortex distance is $r_0/\xi \sim \sqrt{H_{c2}/B}$. The limit of isolated vortices corresponds to $B \ll H_{c2}$ which can be realized if $\kappa \gg 1$.

An isolated vortex is axially symmetric. It has a phase which changes by $2\pi$ after rotation around its axis which we choose as the $z$ axis. We take $\chi = \varphi$ where $\varphi$ is the azimuthal angle in the cylindrical frame $(r,\, \varphi,\, z)$. We thus assumed a cylindrical symmetry of the vortex and can look for a solution in the form

$$\Delta = \Delta_{\mathrm{GL}} f(r) e^{i\varphi}$$

The vector potential has only a $\varphi$-component: $\mathbf{A} = (0,\, A_\varphi,\, 0)$. We have for $f$

$$\xi^2 \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4e^2 Q^2}{\hbar^2 c^2} \right) f + f - f^3 = 0 \tag{133}$$

The gauge invariant vector potential in our case is $\mathbf{Q} = (0,\, A_\varphi - \hbar c/2er,\, 0)$.

Equation (101) becomes

$$\lambda_L^2 \mathrm{curl}\;\mathrm{curl}\, \mathbf{A} + f^2 \mathbf{Q} = 0 \tag{134}$$

For $r \neq 0$ it is

$$\lambda_L^2 \nabla^2 \mathbf{Q} - f^2 \mathbf{Q} = 0 \tag{135}$$

or

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} - \frac{Q}{r^2} - \frac{f^2 Q}{\lambda_L^2} = 0 \tag{136}$$

This equation can be solved in the limit $\kappa \gg 1$. Here $\lambda_L \gg \xi$ and one can put $f = 1$ for $r \gg \xi$. Eq. (136) gives

$$Q = -\frac{\hbar c}{2e\lambda} K_1(r/\lambda_L)$$

Here $K_1(z)$ is the Bessel function of first order of an imaginary argument. For $z \ll 1$ the function $K_1(z) = 1/z$, and it decreases exponentially for large $z$:

$$K_1(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp(-z)$$

The constant at $K_1$ is chosen in such a way that $A_\phi = Q + (\hbar c/2er)$ does not diverge for $r \ll \lambda$. The magnetic field is

$$h_z = \mathrm{curl}_z \mathbf{Q} = \frac{1}{r}\frac{\partial (rQ)}{\partial r} = \frac{\hbar c}{2e\lambda_L^2} K_0(r/\lambda_L)$$

Here $K_0(z)$ is the Bessel function of zero order. For $z \ll 1$ it is $K_0(z) = -\ln z$, and it decreases exponentially for large $z$ in the same way as $K_1$. The magnetic field produced by a single vortex decays exponentially for $r \gg \lambda$ and it is

$$h_z \approx \frac{\phi_0}{2\pi\lambda_L^2} \ln \frac{\lambda_L}{r}$$

The logarithm is cut off at $r \sim \xi$ for $r \ll \lambda$.

For $r \ll \lambda$, $Q = -\hbar c/2er$ and the order parameter satisfies the equation

$$\xi^2 \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \right) f + f - f^3 = 0 \tag{137}$$

This equation coincides exactly with the vortex GP equation (59) for $n = 1$. Its solution saturates at the equilibrium value $f = 1$ for $r \gg \xi$, and decreases as $f \propto r$ for $r \to 0$. The behavior at large distances can be found from Eq. (133). Putting $f = 1 - \delta f$ we find

$$\delta f = \frac{2e^2\xi^2}{\hbar^2 c^2} Q^2 = \frac{\xi^2}{2\lambda_L^2} K_1^2\left(r/\lambda_L\right)$$

For $r \ll \lambda$ we have $\delta f = \xi^2/2r^2$.

A vortex has a core, i.e., the region near its axis with the size of the order of $\xi$ where the order parameter is decreased from its value in the bulk; $|\Delta|$ vanishes at the vortex axis. The vortex core is surrounded by persistent currents which decay away from the vortex core at distances of the order of $\lambda_L$ (Fig. 14).

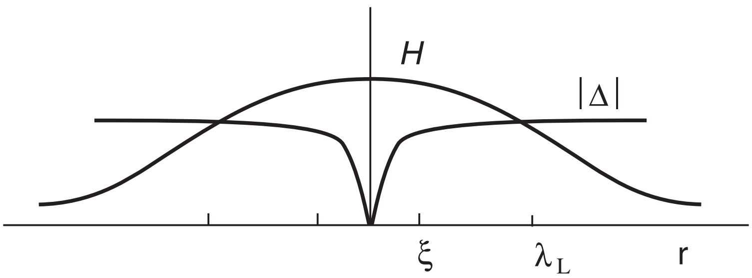


FIG. 14. Structure of a single vortex. The core region with the radius $\xi$ is surrounded by currents. Together with the magnetic field, they decay at distances of the order of $\lambda_L$.

### C. Vortex free energy

Let us calculate the free energy of a single vortex, i.e., the difference between the energy Eq. (70) in the presence of a vortex and the energy without a vortex and the magnetic field.

We have

$$\mathcal{F}_L = \int \left[ \alpha \left( |\Delta|^2 - |\Delta_{\rm GL}|^2 \right) + \frac{\beta}{2} \left( |\Delta|^4 - |\Delta_{\rm GL}|^4 \right) \right. \\ \left. + \gamma \left| \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta \right|^2 + \frac{h^2}{8\pi} \right] dV \tag{138}$$

The kinetic energy term contains $\gamma(4e^2/c^2)Q^2|\Delta|^2$. Since $Q \propto 1/r$ for $\xi \ll r \ll \lambda_L$ this gives a logarithmically large contribution at distances $r \sim \lambda_L$. Simple arguments suggest that we can thus put $|\Delta| = \Delta_{\rm GL}$ everywhere at large distances from the core. The first two terms then vanish, together with the gradients of $\Delta$. The magnetic field gives also a non-logarithmic contribution. As a result, we obtain per unit length of the vortex

$$\mathcal{F}_L = \frac{4e^2\gamma}{c^2} \int Q^2 |\Delta|^2 \, d^2r = \hbar\gamma\Delta_{\rm GL}^2 \ln\left(\frac{\lambda_L}{\xi}\right) \\ = \frac{\Phi_0^2}{16\pi^2\lambda_L^2} \ln\left(\frac{\lambda_L}{\xi}\right) = \left(\frac{\pi\hbar}{m}\right)^2 \frac{mN_s}{4\pi} \ln\left(\frac{\lambda_L}{\xi}\right) \tag{139}$$

The possibility to put $|\Delta| = \Delta_{\rm GL}$ at large distances in Eq. (138) is, however, not so obvious because the function $f$ approaches unity not fast enough, only proportionally to $r^{-2}$. This results in a logarithmically divergence of the terms $|\Delta|^2 - \Delta_{\rm GL}^2$ and $|\Delta|^4 - \Delta_{\rm GL}^4$ in Eq. (138). One can check however, that these two terms cancel each other because the extra factor $\frac{1}{2}$ in front of the second term is compensated by a two times larger power of $\Delta$ as compared to the first term.

For an $n$-quantum vortex we will obtain

$$\mathcal{F}_L = \frac{n^2\Phi_0^2}{16\pi^2\lambda_L^2} \ln\left(\frac{\lambda_L}{\xi}\right). \tag{140}$$

The energy is proportional to $n^2$. Therefore, vortices with $n > 1$ are not favorable: The energy of $n$ single-quantum vortices is proportional to the first power of $n$ and is thus smaller than the energy of one $n$-quantum vortex.

Equation (139) allows to find the lower critical magnetic field, i.e., the field $H$ above which the first vortex appears. The free energy of a unit volume of a superconductor with a set of single-quantum vortices is $F_L = n_L\mathcal{F}_L = (B/\Phi_0)\mathcal{F}_L$. The proper thermodynamic potential in an external field $\mathbf{H}$ is the Gibbs free energy $G = F - \mathbf{HB}/4\pi$

$$G = \frac{B\mathcal{F}_L}{\Phi_0} - \frac{BH}{4\pi} = \frac{B\Phi_0}{16\pi^2\lambda_L^2}\ln\left(\frac{\lambda_L}{\xi}\right) - \frac{BH}{4\pi}$$

It becomes negative for $H > H_{c1}$ where

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda_L^2}\ln\left(\frac{\lambda_L}{\xi}\right). \tag{141}$$

Therefore, vortices appear for $H > H_{c1}$.

We note that

$$H_{c1} = H_{c2}\frac{\ln\kappa}{2\kappa^2} = H_c\frac{\ln\kappa}{\sqrt{2}\kappa} \tag{142}$$

i.e., for superconductors with a large $\kappa$, the critical field $H_{c1}$ is considerably lower than $H_{c2}$.

The phase diagram of a type II superconductor is shown in Fig. 15.

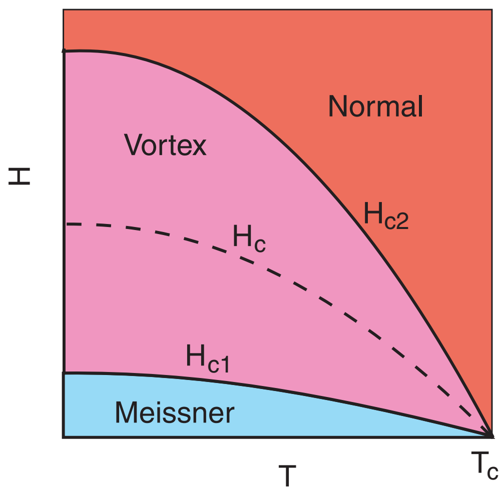


FIG. 15. Phase diagram of a type II superconductor

For more reading on vortices in type II superconductors see Refs.[5,6].

## Problems

### Problem 1

Derive Eq. (132).

### Problem 2

The film with thickness $d \gg \xi$ has the same upper critical field as a bulk superconductor. Find the upper critical field for a film with a thickness $d \ll \xi$ placed in a magnetic field tilted by an angle $\Theta$ from the normal to the film, Fig. 16

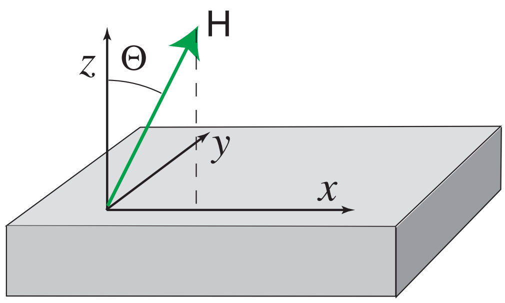


FIG. 16.

## V. LONDON MODEL

The GL equation for the vector potential Eq. (100)

$$\text{curl curl}\,\mathbf{A} + \frac{4\pi N_s e^2}{mc^2}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) = 0 \tag{143}$$

can be extended to a more general case if currents are small and $|\Delta|$ is constant. The temperature needs not to be close to $T_c$. This approximation is good for distances $r \gg \xi$. The latter condition implies that we are restricted to the case of $\kappa \gg 1$.

We assume that the current has a form

$$\mathbf{j} = N_s e \mathbf{v}_s$$

whence $\mathbf{v}_s = \mathbf{j}/N_s e$ and write the free energy as a sum of the kinetic energy of superconducting electrons and the magnetic energy

$$\mathcal{F} = \int \left[\frac{N_s m v_s^2}{2} + \frac{h^2}{8\pi}\right] dV$$

Using the Maxwell equation $\mathbf{j} = (c/4\pi)\text{curl}\,\mathbf{h}$, we transform this to the following form [F. London and H. London, 1935]

$$\mathcal{F} = \frac{1}{8\pi}\int d^3r\ \left[\mathbf{h}^2 + \lambda_L^2\left(\text{curl}\,\mathbf{h}\right)^2\right] \tag{144}$$

Here $\lambda_L$ is

$$\lambda_L = \left(\frac{mc^2}{4\pi N_s e^2}\right)^{\frac{1}{2}} \tag{145}$$

is the London penetration depth. Variation with respect to $\mathbf{h}$ gives the London equation:

$$\mathbf{h} + \lambda_L^2 \text{curl}\,\text{curl}\,\mathbf{h} = 0 \tag{146}$$

this is just a curl of Eq. (143).

## A. Single vortex

Equation (146) holds if there are no vortices. However, the London equation should be modified if there is a vortex. For a vortex we have

$$\mathrm{curl}\,\nabla\chi = 2\pi\hat{\mathbf{z}}\delta^{(2)}(\mathbf{r})$$

where $\hat{\mathbf{z}}$ is the unit vector in the direction of the vortex axis. Therefore, the London equation for a vortex becomes

$$\mathbf{h} + \lambda_L^2 \mathrm{curl}\,\mathrm{curl}\,\mathbf{h} = \mathbf{\Phi}_0 \delta^{(2)}(\mathbf{r}) \tag{147}$$

where $\mathbf{\Phi}_0$ is the vector along the vortex axis with the magnitude of one flux quantum.

For a system of vortices

$$\mathbf{h} + \lambda_L^2 \mathrm{curl}\,\mathrm{curl}\,\mathbf{h} = \mathbf{\Phi}_0 \sum_n \delta^{(2)}(\mathbf{r} - \mathbf{r}_n) \tag{148}$$

where the sum is over all the vortex positions $\mathbf{r}_n$.

If $\mathbf{h} = (0,\, 0,\, h_z(r))$, Eq. (147) gives for $\rho \neq 0$:

$$\lambda_L^2 \left( \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} \right) - h = 0$$

Which gives $h = aK_0(r/\lambda_L)$. The constant can be found from Eq. (147) by integrating it over $d^2r$ over a small area with $r \ll \lambda_L$:

$$\lambda_L^2 \int \mathrm{curl}\,\mathbf{h}\, d\mathbf{l} = \Phi_0$$

so that we obtain $a = \Phi_0/2\pi\lambda_L^2$. Thus

$$h = \frac{\Phi_0}{2\pi\lambda_L^2} K_0 \left( \frac{r}{\lambda_L} \right)$$

It is

$$h(0) = \frac{\Phi_0}{2\pi\lambda_L^2} \ln \left( \frac{\lambda_L}{r} \right)$$

near the vortex axis $r \ll \lambda_L$.

The free energy of the vortex per unit length is

$$\begin{aligned}\mathcal{F} &= \frac{1}{8\pi}\int \left[\mathbf{h}\left(\mathbf{h}+\lambda_L^2 \mathrm{curl\ curl}\,\mathbf{h}\right) + \mathrm{div}[\mathbf{h}\times\mathrm{curl}\,\mathbf{h}]\right]\, d^2r \\ &= \frac{1}{8\pi}\int h_z\Phi_0\delta^{(2)}(r)\, d^2r + \frac{1}{8\pi}\int [\mathbf{h}\times\mathrm{curl}\,\mathbf{h}]\, d\mathbf{l} \end{aligned} \qquad (149)$$

The last integral is taken along a remote contour and vanishes. The first integral gives

$$\mathcal{F}_L = \frac{\Phi_0^2}{16\pi^2\lambda_L^2}\ln\kappa \qquad (150)$$

which is the same as Eq. (140).

### B. Bean–Livingston vortex barrier near the surface of a superconductor

Consider a straight vortex placed near the surface of a superconductor in an applied magnetic field $H$ parallel to the surface. It experiences a force from the applied filed that pushes it into the superconductor since the magnetic field wants to penetrate into the superconductor through this vortex. Another force originates from its interaction with the surface. It is created by an image vortex and attracts our vortex to the surface in the same way as it was for the vortex loop considered earlier in Sect. I.

Let the surface be the $(y, z)$ plane and the magnetic field is along the $z$ axis. Superconductor occupies the half-space $x > 0$.

The magnetic field satisfies the London equation (147), i.e.,

$$\mathbf{h} + \lambda^2 \mathrm{curl\,curl}\,\mathbf{h} = \hat{\mathbf{z}}\Phi_0\delta^{(2)}(\mathbf{r}-\mathbf{r}_L)$$

here $\mathbf{r}$ is two-dimensional coordinate in the $(x, y)$ plane, $\mathbf{r}_L = (x_L, 0)$ is the coordinate of the vortex, and $\hat{\mathbf{z}}$ is the unit vector along the $z$ axis. The boundary conditions at the surface are

$$\mathbf{h} = \mathbf{H},\ [\mathrm{curl}\,\mathbf{h}]_x = (4\pi/c)j_x = 0$$

The magnetic field has the form

$$\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$$

Here $\mathbf{h}_1 = \mathbf{H}\exp(-x/\lambda)$ is due to the decaying external field. The field $\mathbf{h}_2$ is due to the vortex placed near the surface. It is composed by the vortex field and the field created by the image vortex.

$$\mathbf{h}_2 = \mathbf{h}_v + \mathbf{h}_{im}$$

The image vortex is the vortex at the point $(-x_L, 0)$ having the circulation along the $-\hat{\mathbf{z}}$ direction. Its currents flow in the opposite way as compared to the real vortex. As a result, these two vortices produce zero current and

$$\mathbf{h}_2 = 0$$

at the surface (in the middle between the two vortices) thus satisfying both the boundary conditions $j_x = 0$ and $\mathbf{h} = \mathbf{h}_1 = \mathbf{H}$ at the surface together with the London equation in the half-space $x > 0$.

The Gibbs free energy has the form (see Problem 1)

$$\mathcal{G} = \frac{\Phi_0}{4\pi}\left[He^{-x_L/\lambda} + \frac{1}{2}h_2(x_L) - H\right] \tag{151}$$

It vanishes when vortex approaches the surface $x_L = 0$ since $h_2(0) = 0$. For $x_L \gg \xi$, the free energy can also be written as (see Problem 1)

$$\begin{aligned}\mathcal{G} &= \frac{\Phi_0}{4\pi}\left[He^{-x/\lambda} + \frac{\Phi_0}{4\pi\lambda^2}\ln\frac{\lambda}{\xi} - \frac{1}{2}h_v(2x_L) - H\right] \\ &= \frac{\Phi_0}{4\pi}\left[He^{-x/\lambda} - \frac{1}{2}h_v(2x_L) + H_{c1} - H\right]\end{aligned} \tag{152}$$

It is shown in Fig. 17 as a function of the vortex position.

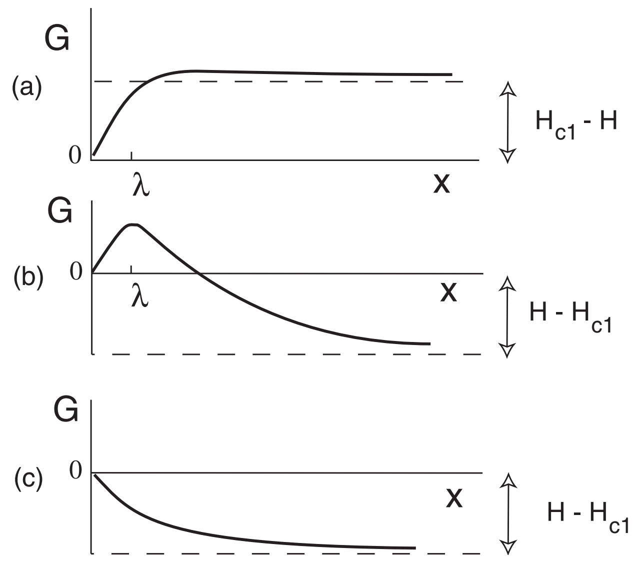


FIG. 17. The free energy of the vortex as a function of its distance from the surface. (a) $H < H_{c1}$. The energy far from the surface is larger. (b) $H_s > H > H_{c1}$. The energy far from the surface is smaller but there is an energy barrier. (c) $H > H_s$. The barrier disappears.

Using

$$h_v(2x_L) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{2x_L}{\lambda}\right)$$

we can calculate the force on the vortex

$$\begin{aligned} F_x = -\frac{\partial \mathcal{G}}{\partial x_L} &= \frac{\Phi_0}{4\pi\lambda}\left[He^{-x/\lambda} + \frac{1}{2}\frac{\partial h_v(2x_L)}{\partial x_L}\right] \\ &= \frac{\Phi_0}{4\pi\lambda}\left[He^{-x/\lambda} - \frac{\Phi_0}{2\pi\lambda^2}K_1\left(\frac{2x_L}{\lambda}\right)\right] \end{aligned}$$

At large distances $x \gg \lambda$ the function $K_1 \propto \exp(-2x_L/\lambda)$ decays faster than the first exponent. The force due to the external field dominates: it tends to push the vortex into the sample. However, the free energy of the vortex in the bulk is lower than the energy at the surface (i.e., zero) only for $H > H_{c1}$ [see Eq. (152)]. This is because the vortex experiences an attraction to the surface at short distances. Indeed, at small distances $x \ll \lambda$, the function $K_1(2x_L/\lambda) = \lambda/2x_L$ and the force

$$F_x = \frac{\Phi_0}{4\pi\lambda}\left[H - \frac{\Phi_0}{4\pi\lambda x_L}\right] \tag{153}$$

is dominated by the image vortex: it attracts the vortex to the surface. Therefore, the vortex energy has a barrier (see Fig. 17).

However, Eq.(153) is only valid when $x \gg \xi$. The force from the image vortex saturates when $x \sim \xi$. Therefore, if the applied field $H > H_s$ where

$$H_s = \frac{\Phi_0}{4\pi\lambda\xi} \sim H_c$$

the force is always positive, barrier vanishes and the vortex can freely penetrate into the superconductor.

### C. System of straight vortices

If the vortices make a regular lattice, one can solve Eq.(148) by means of the Fourier transformation. We introduce

$$h_z = \int e^{i\mathbf{q}\cdot\mathbf{r}} h_q \frac{d^2q}{(2\pi)^2}$$

Here $\mathbf{r}_n$ form a regular lattice with the unit cell vectors $\mathbf{a}$ and $\mathbf{b}$ so that every $\mathbf{r}_n = k\mathbf{a} + m\mathbf{b}$ where $k$ and $m$ are integer numbers, see Fig. 18. In a simple case of a rectangular lattice, $\mathbf{r}_n = (ka,\, mb)$. Equation (148) gives

$$h_q = \frac{\Phi_0}{1 + \lambda_L^2 q^2} \sum_n e^{-i\mathbf{q}\cdot\mathbf{r}_n} \tag{154}$$

The free energy becomes

$$\mathcal{F} = \frac{1}{8\pi} \int \left(1 + \lambda_L^2 q^2\right) h_q h_{-q} \frac{d^2q}{(2\pi)^2} = \frac{\Phi_0^2}{8\pi} \int \frac{1}{(1 + \lambda_L^2 q^2)} \sum_{n,m} e^{i\mathbf{q}(\mathbf{r}_n - \mathbf{r}_m)} \frac{d^2q}{(2\pi)^2} \tag{155}$$

We note that

$$\sum_n e^{i\mathbf{q}\mathbf{r}_n} = (2\pi)^2 n_L \delta^{(2)}(\mathbf{q} - \mathbf{Q}) \tag{156}$$

where $\mathbf{Q}$ is any vector of the reciprocal lattice. They are defined (see Fig. 18) in such a way that $\mathbf{Q} = \mathbf{Q}_1 K + \mathbf{Q}_2 M$ where $K$ and $M$ are integer and

$$\mathbf{Q}_1 = \frac{2\pi[\mathbf{b}\times\hat{\mathbf{z}}]}{([\mathbf{a}\times\mathbf{b}]\cdot\hat{\mathbf{z}})}\ ,\ \mathbf{Q}_2 = -\frac{2\pi[\mathbf{a}\times\hat{\mathbf{z}}]}{([\mathbf{a}\times\mathbf{b}]\cdot\hat{\mathbf{z}})} \tag{157}$$

It can be seen that a dot product of any vector of the real lattice and any vector of the reciprocal lattice is $\mathbf{r}_n\cdot\mathbf{Q} = 2\pi N$ where $N$ is an integer. The area of the unit cell of the reciprocal lattice $A_0 = [\mathbf{Q}_1\times\mathbf{Q}_2]\hat{\mathbf{z}}$ is connected to the area of the unit cell of the real lattice $S_0 = [\mathbf{a}\times\mathbf{b}]\hat{\mathbf{z}}$ by $S_0A_0 = (2\pi)^2$.

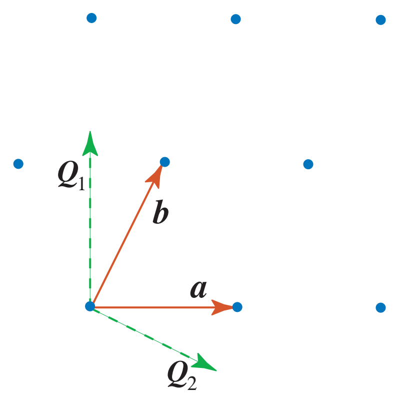


FIG. 18. The vectors of the direct (full lines) and reciprocal (dashed lines) lattices.

Using Eq.(156) we can write

$$\sum_n \int e^{i\mathbf{q}\mathbf{r}_n} f(\mathbf{q})\frac{d^2q}{(2\pi)^2} = \sum_{\mathbf{Q}} n_L f(\mathbf{Q}) \tag{158}$$

Therefore, the free energy becomes

$$\mathcal{F} = \frac{Sn_L\Phi_0^2}{8\pi}\int\frac{1}{(1+\lambda_L^2q^2)}\sum_{n-m}e^{i\mathbf{q}\mathbf{R}_{n-m}}\frac{d^2q}{(2\pi)^2} = \frac{Sn_L^2\Phi_0^2}{8\pi}\sum_Q\frac{1}{1+\lambda_L^2Q^2} \tag{159}$$

where $S$ is the area of the sample. The energy per unit volume becomes

$$F = \frac{B^2}{8\pi}\sum_Q\frac{1}{1+\lambda_L^2Q^2} = \frac{B^2}{8\pi}\left(1+\sum_{Q\neq 0}\frac{1}{1+\lambda_L^2Q^2}\right) \tag{160}$$

To simplify calculation of the sum, we replace it with the integral according to the rule

$$\sum_Q \Rightarrow S_0\int\frac{d^2Q}{(2\pi)^2}$$

where $S_0 = \Phi_0/B$ is the area of the unit cell of the real lattice.

Now we have

$$F = \frac{B^2}{8\pi}\left[1 + S_0\int_{Q>Q_0}^{Q_{max}} \frac{d^2Q}{(2\pi)^2}\frac{1}{1+\lambda_L^2Q^2}\right] = \frac{B^2}{8\pi} + \frac{B\Phi_0}{32\pi^2\lambda_L^2}\ln\left(\frac{\kappa^2}{1+\lambda_L^2Q_0^2}\right) \tag{161}$$

Here $Q_0$ is the first reciprocal-lattice vector in the approximation of a round cell: $S_0\pi Q_0^2 = 4\pi^2$, i.e., $Q_0^2 = 4\pi B/\Phi_0$. We put $Q_{max} = 1/\xi$ since the maximum wave vectors are determined by the shortest distance which is $\xi$ in the London model.

The magnetic induction $B = \Phi_0/S_0$ is determined by the minimum of the Gibbs free energy $F - HB/4\pi$ which gives

$$H = B + \frac{\Phi_0}{8\pi\lambda_L^2}\ln\left(\frac{\kappa^2}{1+\lambda_L^2Q_0^2}\right) - \frac{B}{2(1+\lambda_L^2Q_0^2)} \tag{162}$$

Magnetization $4\pi M = B - H$ becomes a logarithmic function

$$M = -\frac{\Phi_0}{32\pi^2\lambda_L^2}\ln\left(\frac{\kappa^2}{\lambda_L^2Q_0^2}\right) = -\frac{\Phi_0}{32\pi^2\lambda_L^2}\ln\frac{H_{c2}}{B} \tag{163}$$

for $\lambda_L Q_0 \gg 1$. On the other hand,

$$\frac{\partial B}{\partial H} \to \infty \tag{164}$$

for $B \to 0$ as it follows from Eq. (162). The magnetization of a type II superconductor is shown in Fig. 19.

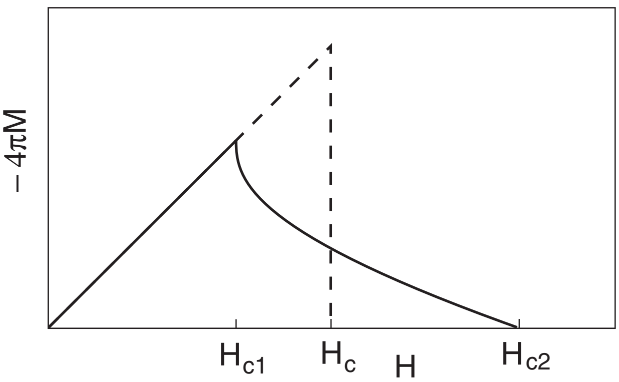


FIG. 19. Full line: Magnetization of a type II superconductor. The linear part at low fields corresponds to the full Meissner effect Eq. (114). The vertical slope at $H = H_{c1}$ is determined by Eq. (164). The linear part near $H_{c2}$ follows from Eq. (131). The magnetization for fields $H_{c1} < H < H_{c2}$ is logarithmic as a function of $H$, Eq. (163). Dashed line: Magnetization of a type I superconductor. The Meissner effect persists up to the thermodynamic critical field $H_c$.

### D. Uncharged superfluids

There is only one known uncharged Fermi superfluid, namely, $^3$He. Its order parameter has a $p$-wave pairing symmetry. Therefore, the above GL model does not directly apply. However, it is possible to construct the corresponding Ginzburg–Landau energy functional and derive the GL equations which are quite similar to those discussed above. We refer the reader to Ref.[22] for more detail on the GL theory for superfluid $^3$He.

In this section we discuss only some selected topics. We note first that the mean-field GL expansion for $^3$He has a wide range of applicability in the same way as for superconductors, see Section III C. Here again it relies on the existence of a large parameter $p_F\xi_0/\hbar \sim 10^2 \div 10^3$ of the same order of magnitude as in superconductors. According to the Ginzburg criterion Eq. (109) the fluctuation region near $T_c$ is thus very narrow; it is practically out of range of the present-day experiment.

Let us consider some aspects relevant to rotating superfluids. In an uncharged superfluid, the London screening length $\lambda_L = \infty$. Vortices in uncharged superfluids are created by *rotating the container* with a superfluid. According to the general thermodynamic principles, for a rotating fluid one has to minimize the combination $\mathcal{F} - \mathbf{L}\cdot\mathbf{\Omega}$ where $\mathbf{L} = \delta\mathcal{F}/\delta\mathbf{\Omega}$ is the angular momentum of the superfluid. One finds

$$\begin{aligned}\mathcal{F} - \mathbf{L}\cdot\mathbf{\Omega} &= \mathcal{F} - [\mathbf{\Omega}\times\mathbf{r}]\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}_s} \\ &= \mathcal{F}\left(\mathbf{v}_s - \mathbf{\Omega}\times\mathbf{r}\right) - \frac{m}{2}\int[\mathbf{\Omega}\times\mathbf{r}]^2\,|\Psi|^2\,dV\end{aligned}$$

The last term should be combined with the contribution from the normal component after which it becomes a constant solid-body rotation energy.

The first term suggests that, in a Bose system where superfluidity is carried out by individual atoms, the momentum operator in the corresponding order parameter equation, like in Eq. (48), should be replaced with

$$-i\hbar\nabla - m\mathbf{\Omega}\times\mathbf{r}$$

For Fermi superfluids, where pairs of particles are superfluid, the comparison of the first term to the GL free energy Eq. (70), (107) with the help of the expression for the superconducting velocity Eq. (96) yields that the vector potential is equivalent to the rotation velocity with the substitution rule

$$\frac{2e}{c}\mathbf{A} \to 2m\,\boldsymbol{\Omega} \times \mathbf{r}$$

A magnetic field $B$ (which is uniform in space since $= \lambda_L = \infty$) is thus equivalent to a rotation velocity $\Omega = eB/2mc$, which agrees with the Larmor theorem.

Moreover, the definition of superconducting velocity $v_s$ used for superconductors is equivalent to the superfluid velocity measured in the rotating frame

$$\mathbf{v}_s \leftrightarrow \frac{\hbar}{2m}\nabla\chi - \boldsymbol{\Omega} \times \mathbf{r}$$

The results obtained for vortices in superconductors can be, for some cases, generalized to uncharged superfluids (more specifically, to superfluid $^3$He) if one takes the limit $e \to 0$ which implies $\lambda_L \to \infty$. Thus, an uncharged superfluid to some extent is similar to an extreme type II superconductor with a very large $\kappa$. With $e \to \infty$, the electric current, of course, disappears but the particle current $j/e$ remains finite.

For example, the upper critical field is still meaningful: it defines the rotation velocity

$$\Omega_{c2} = \frac{\hbar}{4m\xi^2}$$

for which the vortex cores overlap.

The last expression in Eq. (139) contains the circulation quantum of superfluid velocity around a vortex $\kappa_0 = (\pi\hbar/m)$ and the superfluid mass density $mN_s$. Note that this expression for $\kappa_0$ is two times smaller than the expression Eq. (58) for a Bose gas. The reason is that one should insert the mass $2m$ of a Cooper pair for the mass $m$ in Eq. (58); with this in mind the two expressions for $\kappa_0$ are equivalent. Eq. (139) is suitable for superfluids with vortices. For an uncharged superfluid, $\lambda_L \to \infty$, therefore, the logarithm is cut off at intervortex distances

$$\mathcal{F}_L = \frac{\kappa_0^2 m N_s}{4\pi} \ln\left(\frac{r_0}{\xi}\right) \tag{165}$$

The lower critical field, $H_{c1}$, however, corresponds to vanishing rotation velocity: it formally requires the limit $\lambda_L \to \infty$. However, in a finite-size container, the container radius $R$ has to be taken instead of $\lambda_L$. The lower critical field Eq. (141) corresponds now to the critical rotation velocity for which a first vortex becomes favorable in the center of the container. It takes the form[7] (see Problem 6 to Sect. I)

$$\Omega_{c1} = \frac{\hbar}{2mR^2} \ln\left(\frac{R}{\xi}\right)$$

where $R$ is the container radius.

## Problems

### Problem 1

The vortex line is placed parallel to the plane surface of the superconductor. Calculate its energy in an applied magnetic field parallel to the surface (and to the vortex).

## VI. TIME-DEPENDENT GINZBURG–LANDAU THEORY

The simplest theory which describes dynamics of superconductors is the so called time-dependent Ginzburg-Landau theory (TDGL). It generalizes the usual GL theory to include relaxation processes for nonequilibrium superconductivity. Unlike for its static counterpart, the validity of the TDGL theory in a strict sense is much more limited. It is not enough just to have temperatures close to the critical temperature. One also needs that relaxation processes are fast. For real superconductors it is, in general, a very strong limitation which can be fulfilled only for the so called gapless superconductivity. The latter corresponds to a situation where mechanisms working to destroy Cooper pairs are almost successful: the energy gap in the spectrum disappears, but the order parameter and supercurrent still exist. These mechanisms are: interaction with magnetic impurities which act differently on the electrons with opposite spins in a Cooper pair, inelastic (not conserving energy) interactions with phonons, etc. For example, inelastic scattering by phonons is characterized by mean free time $\tau_\epsilon$. The condition for applicability of the TDGL theory is $\tau_\epsilon \Delta \ll \hbar$. Due to a comparatively large magnitude of $\tau_\epsilon$ (for example, $\tau_\epsilon \sim 10^{-9}$ s for Al), this condition is only satisfied in a very narrow vicinity of $T_c$. We will not go into microscopic details here. A justification of using TDGL is that it is simple, gives a generally reasonable picture of superconducting dynamics, and in some cases it can even be derived from the microscopic theory[23].

Time-dependent Ginzburg-landau theory is constructed in the following way[24,25]. In equilibrium, the full energy of a superconductor Eq. (71) has a minimum with respect to $\Delta$ and $\mathbf{A}$:

$$\frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} = 0\,; \ \frac{\delta \mathcal{F}_{sn}}{\delta \mathbf{A}} + c^{-1}\mathbf{j} = 0 \tag{166}$$

where the superconducting energy is determined by Eq. (70)

If the system is out of equilibrium, the order parameter relaxes to its equilibrium value. The rate of relaxation depends on the deviation from equilibrium:

$$-\Gamma \frac{\partial \Delta}{\partial t} = \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \tag{167}$$

where $\Gamma$ is a positive constant. This equation is, however, not gauge invariant. Remember that according to the gauge invariance, the electromagnetic potentials can be changed as

$$\mathbf{A} \to \mathbf{A} + \frac{\hbar c}{2e}\nabla f;\ \phi \to \phi - \frac{\hbar}{2e}\frac{\partial f}{\partial t} \tag{168}$$

without changing the electromagnetic fields

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi;\ \mathbf{h} = \mathrm{curl}\,\mathbf{A} \tag{169}$$

In superconductors, the gauge transformation Eq. (168) is accompanied by the transformation of the order-parameter phase

$$\chi \to \chi + f \tag{170}$$

To preserve the gauge invariant property of equation (167) we have to add there the scalar potential in the form

$$-\Gamma\left(\frac{\partial \Delta}{\partial t} + \frac{2ie\phi}{\hbar}\Delta\right) = \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \tag{171}$$

The second equilibrium condition Eq. (166) is generalized in the following way. Instead of total current in Eq. (166) we put there the supercurrent $\mathbf{j}_s = \mathbf{j} - \mathbf{j}_n$ which accounts for the fact that a part of current can be produced by normal electrons in presence of nonequilibrium electric field:

$$\frac{1}{c}(\mathbf{j} - \mathbf{j}_n) = -\frac{\delta \mathcal{F}_{sn}}{\delta \mathbf{A}} \tag{172}$$

where the normal current is $\mathbf{j}_n = \sigma_n \mathbf{E}$. The electric field is

$$\begin{aligned}\mathbf{E} &= \left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) \\ &= \left(-\frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t} - \nabla\Phi\right)\end{aligned} \tag{173}$$

We introduce here the scalar gauge invariant potential $\Phi$ in addition to the vector potential $\mathbf{Q}$ defined by Eq. (92) so that the full set of gauge invariant potentials becomes

$$\mathbf{Q} = \mathbf{A} - \frac{\hbar c}{2e}\nabla\chi \; ; \; \Phi = \phi + \frac{\hbar}{2e}\frac{\partial\chi}{\partial t} \tag{174}$$

The second line in Eq. (173) was obtained by a gauge transformation using the order parameter phase as a generating function. This can be done if there are no moving vortices in the superconductor: Otherwise, the time and spatial derivatives of the phase do not commute. We shall discuss this effect later.

As we know, the vector potential $\mathbf{Q}$ is related to the superconducting velocity

$$\mathbf{Q} = -\frac{mc}{e}\mathbf{v}_s \tag{175}$$

The physical meaning of the scalar potential $\Phi$ can be seen from Eq. (173). According to it,

$$\frac{\partial \mathbf{v}_s}{\partial t} = \frac{e}{m}\left(\mathbf{E} + \nabla\Phi\right) \tag{176}$$

Since the electric field already includes the gradient of the scalar potential $\phi$, the quantity

$$-(e/m)\Phi = -\frac{\hbar}{2m}\frac{\partial\chi}{\partial t} - (e/m)\phi$$

is the difference between the chemical potential per unit particle mass

$$\mu_s = -\frac{\hbar}{2m}\frac{\partial\chi}{\partial t} \tag{177}$$

of the superconducting particles (i.e., of each of two electrons in a Cooper pair) and the chemical potential

$$\mu_n = (e/m)\phi$$

of the normal particles. Note that these chemical potentials are both measured from a constant chemical potential of the normal state without the electric field, i.e., from the Fermi energy $E_F$ per particle mass.

Note that Eq. (176) coincides with Eq. (7) used in the two-fluid description of superfluids if we put $\mathbf{A} = 0$ for an uncharged superfluid and neglect small terms of the second order in $v_s$. The term $v_s^2$ is indeed small in superconductors: Its maximum possible magnitude is of the order $\hbar^2/m^2\xi^2 \sim v_{\max}^2 \sim \Delta^2/p_F^2$ while $(\hbar/m)\partial\chi/\partial t \sim \hbar|\alpha|/m\Gamma$. We shall see that $\hbar|\alpha|/\Gamma \sim \Delta^2/T_c$. Therefore, the ratio of these two terms is $\sim T_c/E_F \ll 1$.

Finally, the TDGL equations take the form

$$-\Gamma\left(\frac{\partial\Delta}{\partial t}+\frac{2ie\phi}{\hbar}\Delta\right)=-|\alpha|\Delta+\beta|\Delta|^2\Delta-\gamma\hbar^2\left(\nabla-\frac{2ie}{\hbar c}\mathbf{A}\right)^2\Delta, \tag{178}$$

and

$$\mathbf{j}=\mathbf{j}_n+\mathbf{j}_s \tag{179}$$

where, as before, the supercurrent is (compare with Eq. (80))

$$\mathbf{j}_s=-\frac{4e^2\gamma}{c}\left[\Delta^*\left(\mathbf{A}+\frac{i\hbar c}{2e}\nabla\right)\Delta+\Delta\left(\mathbf{A}-\frac{i\hbar c}{2e}\nabla\right)\Delta^*\right]=-\frac{8e^2\gamma}{c}|\Delta|^2\mathbf{Q} \tag{180}$$

Note that it is the total current $\mathbf{j}$ that enters the Maxwell equation (73).

### A. Microscopic values

Microscopic theory justifies the TDGL approach only for dirty gapless superconductors. Therefore, to compare our results with microscopic calculations we use the value for the GL coefficient $\gamma$ as for a dirty superconductor. We remind that

$$\gamma=\nu\pi D/8\hbar T_c \tag{181}$$

where $D=v_F^2\tau/3$ is the diffusion coefficient. Therefore, the relaxation time for current is

$$\tau_j=\frac{\beta\sigma_n}{8e^2\gamma|\alpha|}=\frac{7\zeta(3)\hbar}{4\pi^3T_c}\left(1-\frac{T}{T_c}\right)^{-1} \tag{182}$$

For strong inelastic scattering, the microscopic theory[24,25,23] gives

$$\Gamma=\nu\pi\hbar/8T_c \tag{183}$$

### B. Discussion of TDGL equations

Equation (178) looks like a GP equation with one important difference: The coefficient in front of the time derivative of the wave function $\Delta$ is real in contrast to Eq. (48). This is a crucial distinction: Eq. (178) describes a relaxation dynamics of superconductors. For example, it does not poses a Galilean invariance. The reason is that the heat bath is assumed to be in equilibrium, i.e., at rest in the laboratory frame. It makes a preferable reference frame violating the Galilean invariance.

Equations (178) and (173, 179, 180) define two characteristic relaxation times. Separating real and imaginary parts of Eq. (178) we obtain for the real part

$$-\Gamma\frac{\partial|\Delta|}{\partial t} = \alpha|\Delta| + \beta|\Delta|^3 - \gamma\hbar^2\left(\nabla^2|\Delta| - \frac{4e^2}{\hbar^2c^2}Q^2|\Delta|\right)$$

or

$$\tau_\Delta\frac{\partial|\Delta|}{\partial t} = |\Delta| - \frac{|\Delta|^3}{\Delta_0^2} + \xi^2\left(\nabla^2 - \frac{4e^2}{\hbar^2c^2}Q^2\right)|\Delta| \tag{184}$$

where we introduce the order parameter relaxation time

$$\tau_\Delta = \frac{\Gamma}{|\alpha|} \propto \left(1 - \frac{T}{T_c}\right)^{-1} \tag{185}$$

The equation for the total current

$$\mathbf{j} = -\sigma\left(\frac{1}{c}\frac{\partial\mathbf{Q}}{\partial t} + \nabla\Phi\right) - \frac{8e^2\gamma|\Delta|^2\mathbf{Q}}{c} \tag{186}$$

determines the relaxation time for the vector potential (or current)

$$\tau_j = \frac{\beta\sigma_n}{8e^2\gamma|\alpha|} = \frac{\sigma_n}{8e^2\gamma\Delta_{\mathrm{GL}}^2} \propto \left(1 - \frac{T}{T_c}\right)^{-1} \tag{187}$$

One can define the ratio of these two relaxation times

$$u = \tau_\Delta/\tau_j \tag{188}$$

which is independent of $T$.

For the microscopic value of $\Gamma$, the order-parameter relaxation time is

$$\tau_\Delta = \frac{\pi\hbar}{8T_c}\left(1 - \frac{T}{T_c}\right)^{-1}$$

Ratio of the two times is $u = \pi^4/14\zeta(3) \approx 5.79$.

The imaginary part of Eq. (178) gives

$$c\tau_\Delta|\Delta|^2\Phi + \xi^2\nabla\left(\mathbf{Q}|\Delta|^2\right) = 0 \tag{189}$$

Eq. (189) can also be obtained by calculating the variation of the free energy with respect to the order parameter phase

$$\begin{aligned}
2\frac{\delta\mathcal{F}_{sn}}{\delta\chi} &\equiv i\left(\Delta\,\frac{\delta\mathcal{F}_{sn}}{\delta\Delta} - \Delta^*\,\frac{\delta\mathcal{F}_{sn}}{\delta\Delta^*}\right) \\
&= -i\hbar\gamma\left[\Delta\left(\nabla + \frac{2ie}{\hbar c}\mathbf{A}\right)^2\Delta^* - \Delta^*\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)^2\Delta\right] \\
&= i\hbar^2\gamma\,\mathrm{div}\,\left[\Delta^*\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)\Delta - \Delta\left(\nabla + \frac{2ie}{\hbar c}\mathbf{A}\right)\Delta^*\right] \\
&= -\frac{\hbar}{2e}\,\mathrm{div}\,\mathbf{j}_s
\end{aligned} \tag{190}$$

With help of Eq. (171) we obtain from Eq. (190)

$$\frac{\delta\mathcal{F}_{sn}}{\delta\chi} = -\Gamma|\Delta|^2\left(\frac{\partial\chi}{\partial t} + \frac{2e\phi}{\hbar}\right) = -\frac{\hbar}{4e}\,\mathrm{div}\,\mathbf{j}_s \tag{191}$$

Comparing Eqs. (180) and (191) we obtain Eq. (189).

### C. Energy balance

The TDGL equations support the energy balance equation[24]. Calculate the time derivative of the total free energy, $\mathcal{F}_{sn}$ plus the magnetic energy we obtain the energy balance equation

$$\frac{\partial\mathcal{F}_{tot}}{\partial t} + \int \mathrm{div}\,\mathbf{j}_{\mathcal{F}} = -\int W\,dV \tag{192}$$

where the free energy current is

$$\mathbf{j}_{\mathcal{F}} = -\gamma\hbar^2\left[\left(\frac{\partial\Delta}{\partial t} + \frac{2ie\phi\Delta}{\hbar}\right)\left(\nabla + \frac{2ie}{\hbar c}\mathbf{A}\right)\Delta^* + c.c.\right] + \frac{c}{4\pi}[\mathbf{E}\times\mathbf{h}] \tag{193}$$

and the dissipation function is

$$W = 2\Gamma \left| \left( \frac{\partial \Delta}{\partial t} + \frac{2ie\phi\Delta}{\hbar} \right) \right|^2 + \sigma_n \mathbf{E}^2 \tag{194}$$

It is a positively definite function consisting of two contributions. First comes from the energy loss due to relaxation of the order parameter while the second contribution is associated with the Joule heating by the normal currents.

### D. Charge neutrality and a dc electric field

In metals, all bulk charges are screened. It is seen from the Poisson equation

$$\operatorname{div} \mathbf{E} = -\nabla^2 \phi = 4\pi e (N_e - N_i)$$

where $N_e$ and $N_i$ are densities of electrons and ions, respectively. In equilibrium, $N_e = N_i$. Introduction of an additional electronic charge results in a shift of the chemical potential $\delta\mu = -e\delta\phi$ such that

$$\delta N = N_e - N_i = \frac{\partial N_e}{\partial \mu} \delta\mu = -2\nu e \delta\phi$$

Here $2\nu$ is the density of states, i.e., the number of states within the unit energy interval. The factor 2 accounts for two spin projections of an electron. The Poisson equation becomes

$$\nabla^2 \delta\phi = 8\pi e^2 \nu \delta\phi$$

It demonstrates that the potential $\phi$ decays at distances of the order of the Debye screening length

$$\lambda_D = \left( 8\pi e^2 \nu \right)^{-1/2}$$

which is of the order of interatomic distance in good metals. This result means that variations in the electronic charge density are practically zero at distances of the order of the coherence length, and we have to put the constraint

$$\delta N_e = 0$$

or

$$\operatorname{div} \mathbf{j} = 0 \tag{195}$$

which follows from the continuity equation

$$\frac{\partial(eN_e)}{\partial t} + \operatorname{div} j = 0$$

Eq. (191) now gives

$$\frac{2e\Gamma|\Delta|^2}{\hbar}\Phi = -\frac{\hbar}{4e}\operatorname{div} \mathbf{j}_n = \frac{\hbar\sigma_n}{4e}\left(\nabla^2\Phi + \frac{1}{c}\nabla\frac{\partial \mathbf{Q}}{\partial t}\right) \tag{196}$$

or

$$l_E^2\left(\nabla^2\Phi + \frac{1}{c}\nabla\frac{\partial \mathbf{Q}}{\partial t}\right) = \frac{|\Delta|^2}{\Delta_0^2}\Phi \tag{197}$$

This equation defines the new characteristic length

$$l_E = \left(\frac{\hbar\sigma_n}{8e^2\Gamma\Delta_0^2}\right)^{\frac{1}{2}} = \xi\left(\frac{\tau_j}{\tau_\Delta}\right)^{\frac{1}{2}} \tag{198}$$

It is the electric-field penetration length, i.e., the distance over which an d.c. electric field decays into a superconductor.

In the bulk of a superconductor, in case of equilibrium, we always should have

$$\Phi = \phi + \frac{\hbar}{2e}\frac{\partial\chi}{\partial t} = 0 \tag{199}$$

which is the famous Josephson relation. It shows that the superconducting chemical $\mu_s$ defined by Eq. (177) is equal to the normal chemical potential $\mu_n = (e/m)\phi$ in equilibrium. Remember that we have used this condition in the two-fluid description of superfluidity.

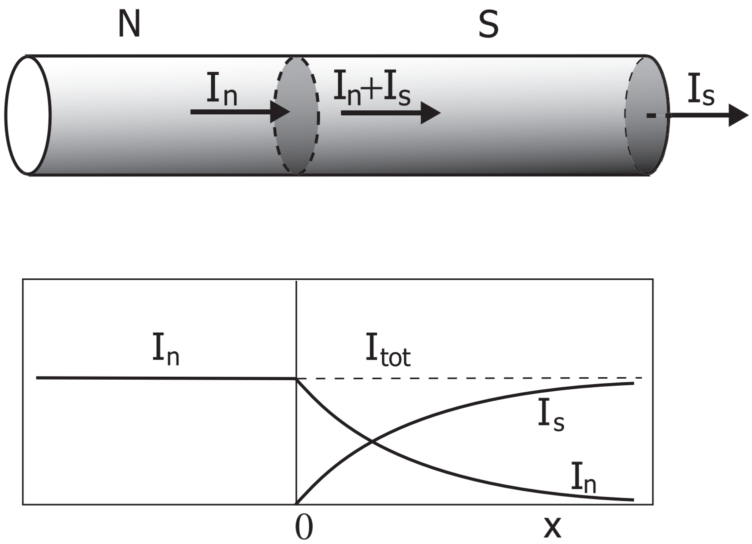

FIG. 20. A boundary between normal metal and superconductor. Normal current converges into supercurrent at distances $l_E$ while the total current is constant.

Consider an interface between a normal metal and a superconductor, and assume that the superconductor occupies the space $x > 0$ (Fig. 20). If there is a current along the $x$ axis flowing from the normal metal into the superconductor, the electric field $E = j/\sigma_n$ exists in the normal metal. However, deep in the superconductor, a d.c. electric field can not exist: It would produce continuous acceleration of superconducting electrons and destroy superconductivity. Indeed, Eq. (197) describes relaxation of $E$. For a constant-in-time electric field and for $\Delta = \Delta_{\rm GL}$

$$\nabla^2\Phi = l_E^{-2}\Phi$$

whence

$$\Phi = El_E \exp(-x/l_E) \tag{200}$$

The constant is chosen to satisfy continuity of current through the interface. According to this equation, electric field together with normal current decay into the superconductor, while supercurrent increases until $j_s$ takes all the current which flows into the superconductor. Normal current

$$j_n = \sigma_n E \exp(-x/l_E)$$

converges into supercurrent over the distance $l_E$.

### E. An a.c. electric field

This consideration refers to an electric field constant in time, the so called longitudinal field $\operatorname{div}\mathbf{E} \neq 0$. An oscillating, transverse electric field, $\operatorname{div}\mathbf{E} \equiv 0$, behaves differently. An example is a radio-frequency electromagnetic wave with a frequency $\omega$ incident on a surface of a superconductor (see Fig. 21). In this case, the current in the superconductor is

$$\mathbf{j} = -\sigma\left(\frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t} + \nabla\Phi\right) - \frac{c}{4\pi\lambda_L^2}\mathbf{Q}$$

Eq. (197) can be satisfied with $\Phi = 0$. The electric field decays into the bulk at distances of the order of (see Problem 1 to Sect. VII)

$$\frac{1}{\lambda_{eff}^2} = \left(\frac{1}{\lambda_L^2} + \frac{1}{\lambda_{skin}^2}\right)$$

where

$$\lambda_{skin} = (1+i)\frac{c}{\sqrt{8\pi\sigma\omega}}$$

is the skin depth in the normal state.

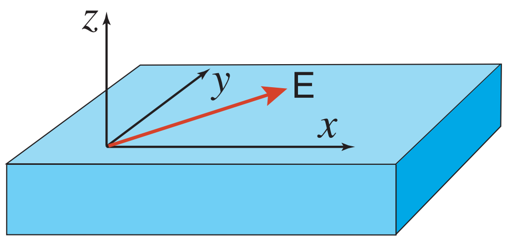


FIG. 21. A radio-frequency wave incident on the superconductor.

If $\omega\tau_\Delta \ll 1$ one has $\lambda_{eff} = \lambda_L$, and the field is expelled due to the Meissner effect. On the contrary, if $\omega\tau_\Delta \gg 1$ one has $\lambda_{eff} = \lambda_{skin}$. The penetration is determined the normal-state skin effect.

### F. Critical current

Consider a thin wire with a radius smaller than $\xi$ (see Fig. 22) and find the maximum current which can flow without dissipation. This problem is exactly the same as the one considered earlier for the GP equation.

FIG. 22. Thin current-carrying superconducting wire

We assume the order parameter in the form

$$\Delta = \Delta_0 f e^{ikx/\xi}$$

The supercurrent takes the form

$$j = j_{GL} f^2 k \tag{201}$$

where $j_{GL}$ is defined by Eq. (105). The order parameter equation (184) becomes

$$\tau_\Delta \frac{\partial f}{\partial t} = f - f^3 - \frac{j^2}{j_{GL}^2} \frac{1}{f^3} \tag{202}$$

The time independent solution is

$$\frac{j}{j_{GL}} = f^2 \sqrt{1 - f^2} = k \left(1 - k^2\right) \tag{203}$$

In the last part of this equation, Eq. (201) is used. As a function of $f$, the current has a maximum $j_{max}/j_{GL} = 2/(3\sqrt{3})$ at $f = \sqrt{2/3}$ or $k = 1/\sqrt{3}$ which coincides with Eq. (63).

Let us study the stability of this solution. We put $f = f_0 + f_1$ where $f_0$ satisfies Eq. (203). We find for the deviation $f_1$ by linearizing Eq. (202)

$$\tau_\Delta \frac{\partial f_1}{\partial t} = 2\left(2 - 3f_0^2\right) f_1 = -2 f_1 \frac{\partial \left(j/j_{GL}\right)}{\partial k} \tag{204}$$

where we express the function $f_0$ in terms of $k$ using Eq. (203). We can see that the time independent solution Eq. (203) is stable for $f_0 > \sqrt{2/3}$ or $1 < k < \sqrt{1/3}$. Equivalently, the solution is stable when

$$\frac{\partial j}{\partial k} > 0$$

(see Fig. 23). The solution in the range $0 < f_0 < \sqrt{2/3}$ or $\sqrt{1/3} < k < 1$ is unstable. With increasing current, the order parameter decreases from $f = 1$ until the minimum value $f = \sqrt{2/3}$ is reached. With further increase in the current, the stability is lost, and superconductivity is destroyed.

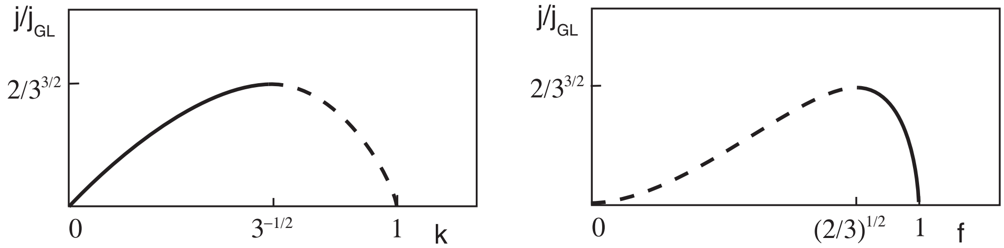


FIG. 23. The supercurrent as a function of $k$ (left panel) or $f$ (right panel). The dashed part of each curve is unstable.

Recall that the problem of the critical current was ambiguous for the GP theory because it did not contain the proper reference frame. For the TDGL theory, such ambiguity does not arise since the reference frame is fixed to be the one where an equilibrium is established in time independent conditions. The situation is quite different here because, in contrast to the GP theory, superconductivity is created on the background of thermal equilibrium at high temperatures with a very efficient relaxation of the order parameter. There is no Galilean invariance because the major part of the fluid consists of the normal component fixed to the laboratory frame. The Landau criterion cannot be used either since the normal excitations do already exist in a stationary system.

## Problems

### Problem 1

Derive Eq. (192).

### Problem 2

Find the resistance of the SN interface for low currents assuming $l_E \gg \xi$.

## VII. MOTION OF VORTICES

TDGL model can be employed to consider the vortex dynamics. Assume that there is an average (transport) current passing through a superconductor with vortices. It exerts a Lorentz force

$$\mathbf{F}_L = \frac{\Phi_0}{c}\left[\mathbf{j}_{\mathrm{tr}} \times \hat{\mathbf{h}}\right] \tag{205}$$

on vortices and pushes them in a direction perpendicular to the current (here $\hat{\mathbf{h}}$ is the unit vector along the magnetic field). The magnitude and direction of the vortex velocity is determined by a balance of the Lorentz force and the forces $\mathbf{F}_{\mathrm{env}}$ acting on a moving vortex from the environment, Fig. 24. In the absence of pinning these forces include friction (longitudinal with respect to the vortex velocity) and gyroscopic (transverse) forces. As a result, vortices move at an angle to the transport current. An electric field perpendicular to the vortex velocity is generated by the moving flux, and a voltage appears across the superconductor with the electric field components both parallel and perpendicular to the current. The longitudinal component produces dissipation in a superconductor while the transverse one is responsible for the Hall effect.

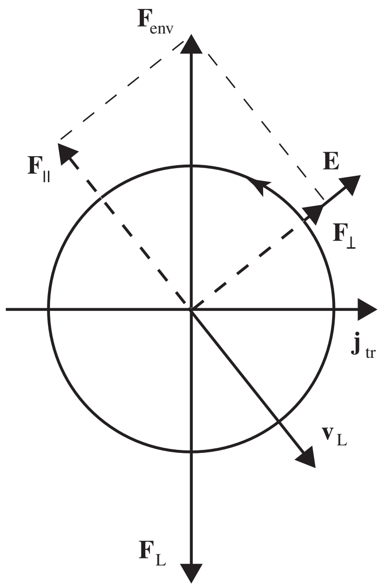


FIG. 24. Forces exerted on a moving vortex.

We already mentioned in Section II that vortices in ideal superfluids at $T = 0$ move with the flow. We shall see now that, in the TDGL model that describes highly dissipative systems, vortices move perpendicular to the flow and generate an electric filed parallel to it, as in a usual dissipative conductor.

### A. A moving vortex and the electric field

The time derivatives $\partial\Delta/\partial t$, $\partial\Delta^*/\partial t$ and $\partial\mathbf{A}/\partial\mathbf{t}$, and the scalar potential $\phi$ can be easily calculated if the vortex velocity $\mathbf{v}_L$ is small. We note that, for a small vortex velocity,

$$\Delta(\mathbf{r}) = \Delta_{st}(\mathbf{r} - \mathbf{v}_L t) + \Delta_1 \ , \ \mathbf{A}(\mathbf{r}) = \mathbf{A}_{st}(\mathbf{r} - \mathbf{v}_L t) + \mathbf{A}_1, \tag{206}$$

where $\Delta_{st}$ and $\mathbf{A}_{st}$ are the values for a stationary vortex, and $\Delta_1$ and $\mathbf{A}_1$ are small corrections. Therefore, within the main terms in the vortex velocity $\mathbf{v}_L$, we can write

$$\frac{\partial\Delta}{\partial t} = -(\mathbf{v}_L\cdot\nabla)\Delta_{st} \ , \ \frac{\partial\mathbf{A}}{\partial t} = -(\mathbf{v}_L\cdot\nabla)\mathbf{A}_{st}. \tag{207}$$

The scalar potential $\varphi$ is to be found from Eq. (189). For small $\mathbf{v_L}$ it contains the order parameter of the stationary vortex and the time derivative of the order parameter phase.

Let us calculate the average electric field. The local electric field is

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi = -\frac{1}{c}\frac{\partial\mathbf{Q}}{\partial t} - \nabla\Phi - \frac{\hbar}{2e}\left(\frac{\partial}{\partial t}\nabla\chi - \nabla\frac{\partial\chi}{\partial t}\right). \tag{208}$$

The last term here is nonzero for a moving vortex. Indeed,

$$\begin{aligned} -\frac{\hbar}{2e}\left(\frac{\partial}{\partial t}\nabla\chi - \nabla\frac{\partial\chi}{\partial t}\right) &= \frac{\hbar}{2e}\left[(\mathbf{v}_L\cdot\nabla)\nabla\chi - \nabla(\mathbf{v}_L\cdot\nabla)\chi\right] \\ &= -\frac{\hbar}{2e}[\mathbf{v}_L\times\mathrm{curl}\,\nabla\chi] \\ &= -\frac{\Phi_0}{c}[\mathbf{v}_L\times\hat{\mathbf{z}}]\mathrm{sgn}(e)\delta^{(2)}(\mathbf{r}). \end{aligned} \tag{209}$$

Here $\hat{\mathbf{z}}$ is the unit vector along the vortex axis in the positive direction of its circulation. The vortex circulation is determined by the sense of the phase increment, it coincides with the magnetic field direction for positive charge of carriers and is antiparallel to it for negative carriers. Equation (208 can be written as

$$\frac{m}{e}\frac{\partial \mathbf{v}_s}{\partial t} = \mathbf{E} + \nabla\Phi + \frac{\Phi_0}{c}[\mathbf{v}_L \times \hat{\mathbf{z}}]\mathrm{sgn}(e)\delta^{(2)}(\mathbf{r}) \tag{210}$$

This equation generalizes Eq. (176) for the case of moving vortices carrying a magnetic flux.

We now average Eq. (208) over the vortex array. Since

$$\frac{\partial \mathbf{Q}}{\partial t} = -(\mathbf{v}_L \cdot \nabla)\mathbf{Q}_{st}$$

the average of this term vanishes because the gauge-invariant vector potential $\mathbf{Q}_{st}$ is a bounded oscillating function (periodic for a periodic vortex array). The average of $\nabla\boldsymbol{\Phi}$ vanishes, too, since $\Phi$ is also a bounded oscillating function periodic for a regular vortex lattice. The only nonzero term in Eq. (208) is thus the one that follows from Eq. (209). It gives

$$\langle \mathbf{E} \rangle_L = \frac{1}{c}\mathbf{B} \times \mathbf{v}_L \tag{211}$$

which is the Faraday's law. The average is defined as

$$\langle \ldots \rangle_L = S_0^{-1} \int_{S_0} (\ldots)\, dS \tag{212}$$

where $S_0 = \Phi_0/B$ is the area occupied by each vortex.

### B. Flux flow: Low vortex density

We consider here the limit $\kappa \gg 1$. In the limit of individual vortices for $\kappa \gg 1$ and $H \ll H_{c2}$ we can neglect the magnetic field and the vector potential compared to the gradient of the order parameter.

Let us calculate the dissipative function $W$ in Eq. (194) in a volume occupied by one vortex. We have per unit vortex length

$$\begin{aligned} \int W\, d^2r &= 2\Gamma \int \left| \left( \frac{\partial \Delta}{\partial t} + \frac{2ie\phi\Delta}{\hbar} \right) \right|^2 d^2r + \int \sigma_n \mathbf{E}^2\, d^2r \\ &= 2\Gamma \int \left[ \left( \frac{\partial |\Delta|}{\partial t} \right)^2 + \frac{4e^2}{\hbar^2}|\Delta|^2\Phi^2 \right] d^2r + \int \sigma_n \mathbf{E}^2\, d^2r \end{aligned} \tag{213}$$

Putting $\mathbf{E} = -\nabla\phi$ we rewrite the second integral in the form

$$\int \sigma_n \mathbf{E}^2 \, d^2 r = -\int \sigma_n \mathbf{E} \nabla\phi \, d^2 r = \int \sigma_n \phi \, \mathrm{div}\, \mathbf{E}$$
$$= -2\Gamma \int \frac{4e^2}{\hbar^2} |\Delta|^2 \Phi \phi \, d^2 r$$

Here we omit the surface contribution and make use of Eq. (196). Inserting this into Eq. (213) we obtain

$$\int W \, d^2 r = 2\Gamma \int \left[ \left( \frac{\partial |\Delta|}{\partial t} \right)^2 + \frac{4e^2}{\hbar^2} |\Delta|^2 \Phi \left( \Phi - \phi \right) \right] d^2 r$$
$$= 2\Gamma \int \left[ \left( \frac{\partial |\Delta|}{\partial t} \right)^2 + \frac{2e}{\hbar} |\Delta|^2 \Phi \frac{\partial \chi}{\partial t} \right] d^2 r$$
$$= 2\Gamma \int \left[ \left[ \left( \mathbf{v}_L \cdot \nabla \right) |\Delta| \right]^2 - \frac{2e}{\hbar} |\Delta|^2 \Phi \left( \mathbf{v}_L \cdot \nabla \right) \chi \right] d^2 r \qquad (214)$$

The two terms under the integral in Eq. (214) represent two different dissipation mechanisms working during the vortex motion. The first term is Tinkham's mechanism[8] of relaxation of the order parameter: Due to the motion of a vortex, the order parameter at a given point varies in time which produces a relaxation accompanied by dissipation. The second is the so-called Bardeen and Stephen[26] contribution. It accounts for normal currents flowing through the vortex core.

The order parameter magnitude $|\Delta| = \Delta_{\mathrm{GL}} f(r)$ for a static vortex satisfies Eq. (137). The gradient of $|\Delta|$ has only a component along the radius-vector in the cylindrical frame $(r, \varphi, z)$:

$$\mathbf{v}_L \cdot \nabla |\Delta| = \Delta_{\mathrm{GL}} v_{L\,r} \frac{df}{dr}$$

The gauge invariant potential $\Phi$ obeys Eq. (197) whence

$$\nabla^2 \Phi - l_E^{-2} f^2 \Phi = 0$$

The scalar potential $\phi$ should be finite. Therefore, we require

$$\Phi \to -\frac{\hbar}{2e} \left( \mathbf{v}_\mathbf{L} \cdot \nabla \right) \chi \qquad (215)$$

for $r \to 0$. We put $r = \xi \tilde{r}$ and

$$\Phi = -\frac{\hbar v_{L\,\varphi} \mu}{2e\xi} \tag{216}$$

where $\varphi$ is the azimuthal angle. The function $\mu(\tilde{r})$ satisfies the equation

$$\left(\frac{d^2\mu}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{d\mu}{d\tilde{r}} - \frac{\mu}{\tilde{r}^2}\right) - uf^2\mu = 0. \tag{217}$$

The condition Eq. (215) requires $\mu \to 1/\tilde{r}$ for $\tilde{r} \to 0$. We see from Eqs. (216), (217) that a moving vortex induces a dipole-like scalar potential proportional to the vortex velocity.

We choose the $x$ axis of the Cartesian coordinate frame along the vortex velocity $\mathbf{v}_L$. The integral over the angle $\varphi$ in Eq. (214) can be calculated using

$$v_{L\,r} = v_L \cos\varphi\,,\ v_{L\,\varphi} = -v_L \sin\varphi$$

Equation (214) gives

$$\int W\, d^2r = 2\pi\Gamma\Delta_{\mathrm{GL}}^2 a v_L^2 \tag{218}$$

where

$$a = \int \left[\left(\frac{df}{d\tilde{r}}\right)^2 + \frac{f^2\mu}{\tilde{r}}\right] \tilde{r}\, d\tilde{r}. \tag{219}$$

Equation (218) can be written in terms of the vortex viscosity $\eta$

$$\int W\, d^2r = \eta v_L^2 \tag{220}$$

where

$$\eta = 2\pi\Gamma\Delta_{\mathrm{GL}}^2 a$$

The moving vortex experiences a friction force

$$\mathbf{F}_{\mathrm{frict}} = -\eta \mathbf{v}_L \tag{221}$$

This force is the only (longitudinal) component $\mathbf{F}_{\|} \equiv \mathbf{F}_{\mathrm{frict}}$ of the force $\mathbf{F}_{\mathrm{env}}$ exerted by the environment, see Fig. 24. It counteracts the Lorentz force Eq. (205) produced by the transport current, i.e.,

$$\frac{\Phi_0}{c}\left[\mathbf{j}_{\rm tr}\times\hat{\mathbf{h}}\right]=\eta\mathbf{v}_L \tag{222}$$

We see that the vortex moves perpendicular to the current.

Expressing the vortex velocity through the electric field using Eq. (211) we find

$$\mathbf{j}_{\rm tr}=\sigma_f\left\langle\mathbf{E}\right\rangle_L \tag{223}$$

where the flux flow conductivity is

$$\sigma_f=\frac{2\pi c^2\Gamma\Delta_{\rm GL}^2 a}{B\Phi_0}. \tag{224}$$

With the microscopic values of $\Gamma$ and $\Delta_{\rm GL}$ taken for dirty superconductors, we find

$$\sigma_f=\frac{ua}{2}\sigma_n\frac{H_{c2}}{B}. \tag{225}$$

The same result can be obtained in a different way. The dissipation in a volume occupied by one vortex can be written as

$$\int W\,d^2r=\sigma_f\left\langle\mathbf{E}\right\rangle_L^2 S_0=\sigma_f\left\langle\mathbf{E}\right\rangle_L^2\Phi_0/B=\sigma_f B\Phi_0 v_L^2/c^2$$

where we use

$$\left\langle\mathbf{E}\right\rangle_L^2=B^2v_L^2/c^2$$

Comparing it with Eq. (218) we again arrive at Eq. (224).

To calculate $a$ we need first to solve Eq. (217). For $u=5.79$, one finds $a\approx 0.502$. The flux flow conductivity becomes[27]

$$\sigma_f\approx 1.45\sigma_n\frac{H_{c2}}{B}. \tag{226}$$

It is close to the expression $\sigma_f=\sigma_n\,H_{c2}/B$ known as the Bardeen–Stephen model[26]. This simple prediction implies that the flux flow resistivity is just a normal-state resistivity times the fraction of space occupied by vortex cores.

$$\rho_f=\frac{\rho_n}{1.45}\frac{B}{H_{c2}}\approx 0.69\rho_n\frac{B}{H_{c2}}$$

The flux-flow resistivity is linear in $B$.

### C. Flux flow: High vortex density

Consider now high fields, $H \to H_{c2}$. We put $\mathbf{A} = (0,\, Hx,\, 0)$, and choose the static potential in the form $\phi = -E_x x$, where the electric field $E_x = -(v_y/c)H$ is homogeneous in the leading approximation and has only an $x$ component. The vortex velocity has only a $y$ component $\mathbf{v}_L = (0, v_y, 0)$. Recall that the static order parameter in the vicinity of the upper critical field satisfies the linearized GL equation

$$\frac{\partial^2 \Delta}{\partial x^2} + \left(\frac{\partial}{\partial y} - \frac{2ieHx}{\hbar c}\right)^2 \Delta + \xi^{-2}\Delta = 0 \tag{227}$$

(see Section IV). It has the form Eq. (122):

$$\Delta = \sum_n C_n e^{iqny} \exp\left[-\frac{1}{2\xi^2}\left(x - \frac{\hbar cqn}{2eH}\right)^2\right]. \tag{228}$$

We calculate the average dissipative function Eq. (194):

$$\begin{aligned} \langle W\rangle_L &= 2\Gamma \left\langle \left(\frac{\partial \Delta}{\partial t} - \frac{2ieE_x x}{\hbar}\Delta\right)\left(\frac{\partial \Delta^*}{\partial t} + \frac{2ieE_x x}{\hbar}\Delta^*\right)\right\rangle_L + \sigma_n E_x^2 \\ &= 2\Gamma v_y^2 \left\langle \left(\frac{\partial \Delta}{\partial y} - \frac{2ieHx}{\hbar c}\Delta\right)\left(\frac{\partial \Delta^*}{\partial y} + \frac{2ieHx}{\hbar c}\Delta^*\right)\right\rangle_L + \sigma_n E_x^2 \end{aligned} \tag{229}$$

To calculate the average we make use of the identities

$$\frac{\partial \Delta}{\partial x} = \left(-i\frac{\partial \Delta}{\partial y} - \frac{2eHx}{\hbar c}\Delta\right),\ \frac{\partial \Delta^*}{\partial x} = \left(i\frac{\partial \Delta^*}{\partial y} - \frac{2eHx}{\hbar c}\Delta^*\right), \tag{230}$$

derived for the order parameter Eq. (228).

We have

$$\left\langle \left(\frac{\partial \Delta}{\partial y} - \frac{2ieHx}{\hbar c}\Delta\right)\left(\frac{\partial \Delta^*}{\partial y} + \frac{2ieHx}{\hbar c}\Delta^*\right)\right\rangle_L = \left\langle \frac{\partial \Delta}{\partial x}\frac{\partial \Delta^*}{\partial x}\right\rangle_L .$$

Therefore

$$\begin{aligned} &\left\langle \left(\frac{\partial \Delta}{\partial y} - \frac{2ieHx}{\hbar c}\Delta\right)\left(\frac{\partial \Delta^*}{\partial y} + \frac{2ieHx}{\hbar c}\Delta^*\right)\right\rangle_L \\ &= -\frac{1}{2}\left\langle \Delta\left[{}^*\left(\frac{\partial}{\partial y} - \frac{2ieHx}{\hbar c}\right)^2 \Delta + \frac{\partial^2 \Delta}{\partial x^2}\right]\right\rangle_L = \frac{1}{2\xi^2}\left\langle |\Delta|^2\right\rangle_L . \end{aligned} \tag{231}$$

We use here Eq. (227). We obtain from Eq. (229)

$$\begin{aligned}\langle W\rangle_L &= \Gamma\xi^{-2}v_y^2\left\langle|\Delta|^2\right\rangle_L+\sigma_n E_x^2\\ &=\left(\sigma_n+\frac{\Gamma c^2}{\xi^2H^2}\left\langle|\Delta|^2\right\rangle_L\right)E_x^2=\sigma_f E_x^2\end{aligned} \tag{232}$$

where the flux flow conductivity is

$$\sigma_f=\sigma_n+4e^2\hbar^{-2}\Gamma\xi^2\left\langle|\Delta|^2\right\rangle_L. \tag{233}$$

The average magnitude of the order parameter is from Eq. (130)

$$\left\langle|\Delta|^2\right\rangle_L=\frac{2\kappa^2\Delta_{\mathrm{GL}}^2}{(2\kappa^2-1)\,\beta_A}\left(1-\frac{B}{H_{c2}}\right).$$

Therefore

$$\begin{aligned}\sigma_f &= \sigma_n+\frac{4e^2\hbar^{-2}\Gamma\xi^2\Delta_{\mathrm{GL}}^2}{\beta_A[1-1/(2\kappa^2)]}\left(1-\frac{B}{H_{c2}}\right)\\ &=\sigma_n\left[1+\frac{u}{2}\,\frac{(1-B/H_{c2})}{\beta_A[1-(1/2\kappa^2)]}\right].\end{aligned} \tag{234}$$

The flux-flow resistivity is

$$\rho_f=\rho_n\left[1-\frac{u}{2}\,\frac{(1-B/H_{c2})}{\beta_A[1-(1/2\kappa^2)]}\right]$$

It is linear in $B$. The slope is

$$\left.\frac{\partial\rho_f}{\partial B}\right|_{B=H_{c2}}=\frac{\rho_n}{H_{c2}}\frac{u}{2\beta_A[1-(1/2\kappa^2)]}\approx\frac{2.5}{[1-(1/2\kappa^2)]}\frac{\rho_n}{H_{c2}}$$

The flux-flow resistivity is shown in Fig. as a function of the magnetic induction.

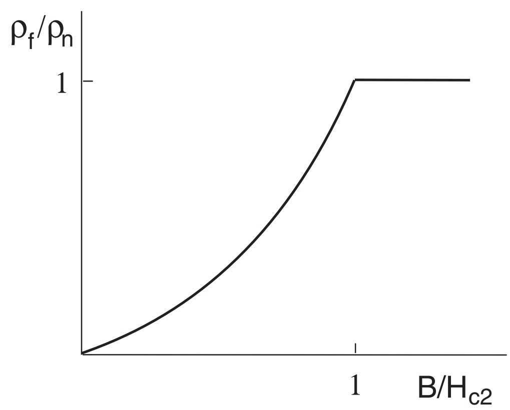


FIG. 25. Flux flow resistivity within the TDGL model as a function of the magnetic field.

## Problems

**Problem 1: additional problem to Sect. VI**

Find the penetration depth for a magnetic field oscillating with a small frequency $\omega$ such that $\mathbf{A} = \mathbf{A}_0 e^{-i\omega t}$.

**Problem 2**

Find the solution of linearized TDGL equation in a uniform electric filed with a potential $\phi = -E_x x$ that describes a moving vortex lattice for $H$ close to $H_{c2}$.

**Problem 3**

Calculate the current for a moving vortex lattice from the previous problem.

## VIII. PARACONDUCTIVITY

A small superconducting order parameter can appear above the transition temperature due to fluctuations. In the presence of an electric field, existence of a nonzero fluctuating order parameter will give rise to a nonzero supercurrent which would enhance the conductivity of the superconductor. This fluctuation-induced additional conductivity above $T_c$ is called the paraconductivity. Also, a nonzero fluctuation magnetization and other superconducting properties can appear already at temperatures above $T_c$. Here we calculate the fluctuation conductivity using the TDGL model. Of course, the applicability of this result is limited by the validity of the TDGL model itself. This result was first obtained by Aslamazov and Larkin in 1968.

Consider the fluctuation-induced order parameter above $T_c$. In the Fourier representation

$$\Delta(\mathbf{r}) = \sum_{\mathbf{p}} \Delta_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}\hbar}$$

The sum over momenta is the sum over the states in a unit volume. In a system with dimensions $L_1$, $L_2$ and $L_3$ the momentum components assume the values $p_x = 2\pi\hbar n_x/L_1$, $p_y = 2\pi\hbar n_y/L_2$, and $p_z = 2\pi\hbar n_z/L_3$ where $n_x, n_y, n_z$ are integer numbers. The sum is

$$\sum_{\mathbf{p}} \equiv (L_1L_2L_3)^{-1} \sum_{n_x,n_y,n_z}$$

Since the order parameter varies over distances of the order of $\xi$, a characteristic scale for $p$ is $\hbar/\xi$. In a three dimensional case, where all $L_1, L_2, L_3 \gg \xi$, the numbers $n_x$, $n_y$, $n_z$ can assume large values. Therefore, in three dimensions, the sum is

$$\sum_{\mathbf{p}}^{(3D)} = (L_1L_2L_3)^{-1} \sum_{n_x,n_y,n_z} = (L_1L_2L_3)^{-1} \int \frac{L_1L_2L_3}{(2\pi\hbar)^3}\, dp_x\, dp_y\, dp_z = \int \frac{d^3p}{(2\pi\hbar)^3}$$

In a film with a thickness $L_1 = d \ll \xi$, the number $n_x$ can only have one value $n_x = 0$. Therefore,

$$\sum_{\mathbf{p}}^{(2D)} = (dL_2L_3)^{-1} \sum_{n_y,n_z} = (dL_2L_3)^{-1} \int \frac{L_2L_3}{(2\pi\hbar)^2}\, dp_y\, dp_z = d^{-1} \int \frac{d^2p}{(2\pi\hbar)^2}$$

Similarly, in the one dimensional case, i.e., for a wire with the cross-section area $S \ll \xi^2$

$$\sum_{\mathbf{p}}^{(1D)} = S^{-1} \int \frac{dp}{2\pi\hbar}$$

In the absence of the electric and magnetic fields, the free energy of the superconductor is, within the leading terms in small $|\Delta|$

$$\begin{aligned} \mathcal{F}_{sn} &= \int \left[\alpha|\Delta|^2 + \gamma\hbar^2 \left|\nabla\Delta\right|^2\right] dV \\ &= \sum_{\mathbf{p}} \left[\alpha + \gamma\mathbf{p}^2\right] |\Delta_{\mathbf{p}}|^2 \end{aligned} \tag{235}$$

Note that $\alpha > 0$ above $T_c$. The probability of fluctuations is

$$\begin{aligned} P\{\Delta\} &= C \exp\left(-\mathcal{F}_{sn}/T\right) \\ &= C \exp\left(-T^{-1} \sum_{\mathbf{p}} \left[\alpha + \gamma\mathbf{p}^2\right] |\Delta_{\mathbf{p}}|^2\right) \\ &= C \prod_{\mathbf{p}} \exp\left(-T^{-1}\left[\alpha + \gamma\mathbf{p}^2\right] |\Delta_{\mathbf{p}}|^2\right) \end{aligned} \tag{236}$$

For each $\mathbf{p}$ we have

$$P\{\Delta_{\mathbf{p}}\} = C_p \exp\left(-T^{-1}\left[\alpha + \gamma\mathbf{p}^2\right] |\Delta_{\mathbf{p}}|^2\right) \tag{237}$$

and $C = \prod_{\mathbf{p}} C_{\mathbf{p}}$. The normalization constant is

$$C_p^{-1} = \int_0^\infty \exp\left(-T^{-1}\left[\alpha + \gamma\mathbf{p}^2\right] |\Delta_{\mathbf{p}}|^2\right) d\left|\Delta_{\mathbf{p}}\right|$$

The average fluctuation is

$$\begin{aligned} \left\langle |\Delta_{\mathbf{p}}|^2 \right\rangle &= C_p \int_0^\infty |\Delta_{\mathbf{p}}|^2 \exp\left(-T^{-1}\left[\alpha + \gamma\mathbf{p}^2\right] |\Delta_{\mathbf{p}}|^2\right) d\left|\Delta_{\mathbf{p}}\right| \\ &= \frac{T}{2\left(\alpha + \gamma\mathbf{p}^2\right)} \end{aligned} \tag{238}$$

Consider the TDGL equation above $T_c$ so that $\alpha > 0$. The order parameter appears due to fluctuations and is thus small. The linearized TDGL equation has the form

$$-\Gamma\left(\frac{\partial\Delta}{\partial t} + \frac{2ie\phi}{\hbar}\Delta\right) = \alpha\Delta - \gamma\hbar^2\nabla^2\Delta \tag{239}$$

with $\phi = -\mathbf{E}\cdot\mathbf{r}$. We put $\Delta = \Delta_{\mathbf{p}}^{(0)} + \Delta_{\mathbf{p}}^{(1)}$ where $\Delta_{\mathbf{p}}^{(0)}$ is the order parameter without the electric field, and $\Delta_{\mathbf{p}}^{(1)}$ is a correction proportional to $E$. For a time-independent state, it satisfies the equation

$$\left(\alpha + \gamma\mathbf{p}^2\right)\Delta_{\mathbf{p}}^{(1)} = \frac{2ie\Gamma}{\hbar}\left(\mathbf{E}\cdot\hat{\mathbf{r}}\right)\Delta_{\mathbf{p}}^{(0)} \tag{240}$$

where the operator $\hat{\mathbf{r}}$ in the Fourier representation is $\hat{\mathbf{r}} = i\hbar\partial/\partial\mathbf{p}$. Therefore,

$$\Delta_{\mathbf{p}}^{(1)} = -2e\Gamma\left(\alpha + \gamma\mathbf{p}^2\right)^{-1}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{p}}\Delta_{\mathbf{p}}^{(0)} \tag{241}$$

The fluctuation-induced supercurrent is

$$\begin{aligned}\langle\mathbf{j}_s\rangle &= 2e\gamma\left\langle\left[\Delta^*\left(-i\hbar\nabla\Delta\right) + \Delta\left(i\hbar\nabla\Delta^*\right)\right]\right\rangle \\ &= 4e\gamma\sum_{\mathbf{p}}\mathbf{p}\left\langle\left|\Delta_{\mathbf{p}}\right|^2\right\rangle \\ &= 4e\gamma\sum_{\mathbf{p}}\mathbf{p}\left\langle\left|\Delta_{\mathbf{p}}^{(0)}\right|^2 + \Delta_{\mathbf{p}}^{*(0)}\Delta_{\mathbf{p}}^{(1)} + \Delta_{\mathbf{p}}^{(0)}\Delta_{\mathbf{p}}^{*(1)}\right\rangle\end{aligned} \tag{242}$$

The sum over momenta of the first term disappears. The correction gives

$$\begin{aligned}\langle\mathbf{j}_s\rangle &= -8e^2\gamma\Gamma\sum_{\mathbf{p}}\frac{\mathbf{p}}{\alpha + \gamma\mathbf{p}^2}\left\langle\Delta_{\mathbf{p}}^{*(0)}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{p}}\Delta_{\mathbf{p}}^{(0)} + \Delta_{\mathbf{p}}^{(0)}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{p}}\Delta_{\mathbf{p}}^{*(0)}\right\rangle \\ &= -8e^2\gamma\Gamma\sum_{\mathbf{p}}\frac{\mathbf{p}}{\alpha + \gamma\mathbf{p}^2}\left(\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{p}}\left\langle\left|\Delta_{\mathbf{p}}^{(0)}\right|^2\right\rangle\right)\end{aligned} \tag{243}$$

Using Eq. (238) we find

$$\langle\mathbf{j}_s\rangle = 8Te^2\gamma^2\Gamma\sum_{\mathbf{p}}\frac{\mathbf{p}\left(\mathbf{p}\cdot\mathbf{E}\right)}{\left(\alpha + \gamma p^2\right)^3} = \frac{\pi e^2 T_c}{\hbar\left(T - T_c\right)}\sum_{\mathbf{p}}\frac{\mathbf{p}\left(\mathbf{p}\cdot\mathbf{E}\right)\xi^4/\hbar^2}{\left(1 + \xi^2p^2/\hbar^2\right)^3} \tag{244}$$

Here we use the microscopic value

$$\Gamma = \frac{\pi\hbar\nu}{8T_c}$$

and the definition of the coherence length $\xi = \hbar\sqrt{\gamma/|\alpha|}$.

Let us calculate the fluctuation supercurrent for a film. In the $2D$ case, Eq. (244) gives

$$\begin{aligned}\langle\mathbf{j}_s\rangle &= \frac{8Te^2\gamma^2\Gamma}{d}\int\frac{\mathbf{p}\left(\mathbf{p}\cdot\mathbf{E}\right)}{\left(\alpha + \gamma p^2\right)^3}\frac{d^2p}{(2\pi\hbar)^2} \\ &= \mathbf{E}\frac{2Te^2\gamma^2\Gamma}{\pi\hbar^2 d}\int_0^\infty\frac{p^3\,dp}{\left(\alpha + \gamma p^2\right)^3} \\ &= \sigma'\,\mathbf{E}\end{aligned} \tag{245}$$

where we use that $d^2p = pdp\, d\varphi_p$ and

$$\int_0^{2\pi} p_i p_k \,\frac{d\varphi_p}{2\pi} = \frac{p^2}{2}\delta_{ik}$$

The excess fluctuation conductivity is

$$\begin{aligned}\sigma' &= \frac{2Te^2\gamma^2\Gamma}{\pi\hbar^2 d}\int_0^\infty \frac{p^3\, dp}{(\alpha+\gamma p^2)^3} = \frac{T_c e^2\Gamma}{\pi\hbar^2 d\alpha}\int_0^\infty \frac{x^3\, dx}{(1+x^2)^3} \\ &= \frac{T_c e^2\Gamma}{2\pi\hbar^2 d\alpha} = \frac{e^2}{16\hbar d}\,\frac{T_c}{T-T_c} = \frac{1}{8R_Q d}\,\frac{T_c}{T-T_c}\end{aligned} \tag{246}$$

Remarkably, the fluctuation conductivity in a film does not depend on the material parameters of the superconductor. We denote here

$$\frac{2\hbar}{e^2} = R_Q \approx 6\ k\Omega$$

It is called the quantum resistance.

In three and one dimensions, the fluctuation conductivity is

$$\sigma' = \begin{cases} e^2T/[32\hbar(T-T_c)\xi] & \text{in } 3D \\ \pi e^2 T_c\xi/[16S\hbar\,(T-T_c)] & \text{in } 1D \end{cases} \tag{247}$$

The fluctuation conductivity diverges as $1/\sqrt{1-T/T_c}$ in the three dimensional case, while it diverges as $(1-T/T_c)^{-3/2}$ in the one-dimensional case.

## Problems

### Problem 1

Calculate the fluctuation conductivity in the three-dimensional case.

### Problem 2

The same for the one-dimensional case.

## IX. WEAK LINKS

### A. Aslamazov–Larkin model

#### 1. D.C. Josephson Effect

Consider two superconductors $S_1$ and $S_2$ separated by an insulating layer with a hole in it with an area $A \ll \xi^2$. The thickness of the insulating layer is $L \ll \xi$. This contact is equivalent to a constriction between two superconductors, the so called superconducting bridge (see Fig. 26). The both configurations belong to the so called superconducting weak links.

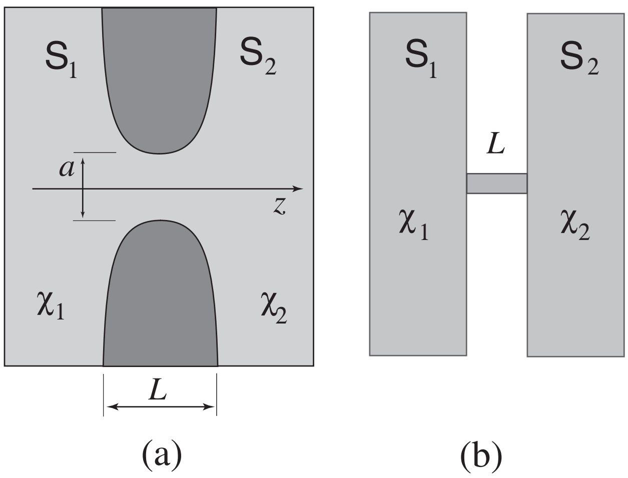


FIG. 26. Examples of superconducting weak links. (a) A hole in an insulating layer between two bulk superconductors $S_1$ and $S_2$. (b) Superconducting bridge between two bulk superconductors.

It is possible to derive a simple equation for the supercurrent through such weak links. The GL equation (90) is

$$\xi^2 \left( \nabla - \frac{2ie}{\hbar c} \mathbf{A} \right)^2 \Delta + \Delta - \Delta |\Delta|^2 / \Delta_{\mathrm{GL}}^2 = 0 \tag{248}$$

For distances of the order of $L$, the gradient terms are of the order $(\xi/L)^2 \Delta$, i.e., they are much larger than all other terms. We obtain the Laplace equation

$$\nabla^2 \Delta = 0$$

Let us take the order parameter in the form

$$\Delta = \Delta_1 e^{i\chi_1} \left[1 - f(\mathbf{r})\right] + \Delta_2 e^{i\chi_2} f(\mathbf{r})$$

The function $f(\mathbf{r})$ also satisfies the Laplace equation

$$\nabla^2 f = 0$$

with the boundary conditions $f = 0$ in the superconductor 1, and $f = 1$ in the superconductor 2.

The supercurrent density Eq. (80) becomes

$$\begin{aligned}
\mathbf{j}_s &= 2e\gamma\hbar\left[-i\Delta^*\nabla\Delta + i\Delta\nabla\Delta^*\right] \\
&= -2ie\gamma\hbar\Delta_1\Delta_2\left[e^{i(\chi_2-\chi_1)} - e^{-i(\chi_2-\chi_1)}\right]\nabla f \\
&= 4e\gamma\hbar\Delta_1\Delta_2 \sin\left(\chi_2 - \chi_1\right)\left(\nabla f\right) \qquad (249)
\end{aligned}$$

Integrating over the cross section of the hole we get the full current along the $z$ axis

$$I_s = I_c \sin\left(\chi_2 - \chi_1\right) \qquad (250)$$

where

$$I_c = 4e\gamma\hbar\Delta_1\Delta_2 \int \left(\nabla_z f\right)\, dS \qquad (251)$$

Note that the function $f$ increases from region 1 into the region 2, therefore, the value $I_c$ is positive. By the order of magnitude,

$$\int \left(\nabla_z f\right)\, dS \sim A/L$$

If the two superconductors are connected by a one-dimensional channel (wire) of a length $L$ and a cross section area $A$, such that the ends correspond to $z = 0$ and $z = L$ the function is $f = z/L$ and we get exactly

$$\int \left(\nabla_z f\right)\, dS = A/L$$

Equation (250) describes the d.c. (stationary) Josephson effect: the supercurrent through a weak link is proportional to a sine of the phase difference. Eq. (250) is called also the current-phase relation. It is sinusoidal for a short weak link with $L \ll \xi$. We shall discuss other realizations of the Josephson effect in the following sections.

### *2. A.C. Josephson Effect*

Using the same arguments one can also calculate the normal current. For a short contact, the gradient term is the largest. Therefore,

$$\mathbf{j}_n = -\sigma_n \nabla \phi$$

and also

$$\operatorname{div} \mathbf{j}_n = 0$$

The potential thus satisfies the Laplace equation

$$\nabla^2 \phi = 0$$

We can put

$$\phi = V\left[1 - f(\mathbf{r})\right]$$

where $V$ is the voltage across the weak link. The function $f$ satisfies the Laplace equation with the same boundary conditions as before; it is therefore the same function as was introduced above. The normal current is

$$\mathbf{j}_n = \sigma_n V \nabla f(\mathbf{r})$$

The full normal current becomes

$$I_n = V/R_n$$

where

$$\frac{1}{R_n} = \sigma_n \int (\nabla_z f) \, dS \sim \sigma_n A/L \tag{252}$$

is the total resistance of the hole in the normal state. For a one-dimensional wire one has exactly $R_n = \sigma_n^{-1} L/A$.

With the general expression Eq. (252) we can write the critical supercurrent Eq. (251) in the form

$$I_c = \frac{4e\gamma\hbar\Delta_1\Delta_2}{\sigma_n R_n} = \frac{\pi\Delta_1\Delta_2}{4T_c e R_n} \tag{253}$$

where we use the expression $\gamma = \pi\nu D/8T_c\hbar$ for dirty superconductors. We see that $I_c$ is now expressed through the resistance of the link in the normal state.

The total current takes the form

$$I = \frac{V}{R_n} + I_c \sin\varphi \tag{254}$$

where the phase difference is $\varphi = \chi_2 - \chi_1$. Deep in the superconducting region, the Josephson equation (199) holds

$$\Phi = \phi + \frac{\hbar}{2e}\frac{\partial\chi}{\partial t} = 0 \tag{255}$$

Since the full current through the small hole is also small, the current density far from the hole is vanishingly small. Thus, the phases $\chi_1$ and $\chi_2$ do not vary far from the hole, $\chi_{1,2} = const$. As a result, the difference of the phases at the both sides from the hole obeys the Josephson relation

$$\hbar\frac{\partial\varphi}{\partial t} = 2eV \tag{256}$$

This equation describes the so called resistively shunted Josephson junction (RSJ) model (see Fig. 27).

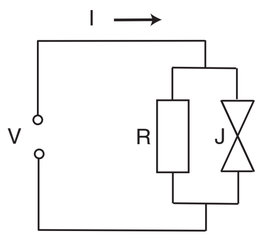


FIG. 27. The resistively shunted Josephson junction.

The full equation for the current is

$$I = \frac{\hbar}{2eR_n}\frac{\partial \varphi}{\partial t} + I_c \sin\varphi \tag{257}$$

If $I < I_c$, the phase is stationary:

$$\varphi = \arcsin(I/I_c)$$

and voltage is zero. the phase difference reaches $\pi/2$ for $I = I_c$. If $I > I_c$, the phase starts to grow with time. Denote $t_0$ the time it takes to increase from $\pi/2$ to $\pi/2 + 2\pi$. The average voltage is

$$(2e/\hbar)\overline{V} = 2\pi/t_0$$

Calculating $t_0$ (see Problem 1) we find the current–voltage curve dependence

$$\overline{V} = R_n\sqrt{I^2 - I_c^2} \tag{258}$$

It is shown in Fig. 28.

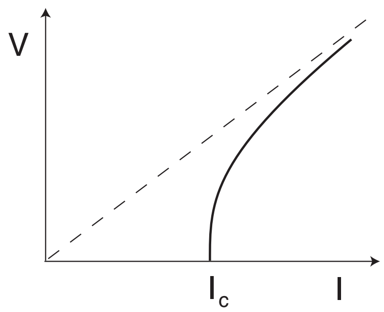


FIG. 28. The current–voltage curve for resistively shunted Josephson junction.

**Problems**

### Problem 1

Derive Eq. (258).

## X. LAYERED SUPERCONDUCTORS.

### A. Lawrence–Doniach model

Consider a stack of thin superconducting films with a thickness smaller than $\xi$ separated by insulating layers of a thickness $d$. Let the $z$ axis be perpendicular to the films (see Fig. 29).

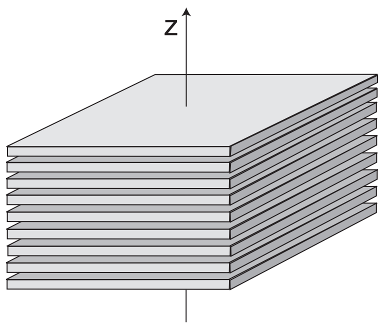


FIG. 29. The model of a layered superconductor.

The free energy can be written as

$$\mathcal{F}_{sn} = d\sum_n \int \left[ \alpha|\Delta_n(\mathbf{r})|^2 + \frac{\beta}{2}|\Delta_n(\mathbf{r})|^4 + \gamma_{\|} \left| \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta_n(\mathbf{r}) \right|^2 + F_{int}(n) \right] d^2r \quad (259)$$

Here the sum is over all the layers; $\Delta_n(\mathbf{r})$ is the value of the order parameter at the layer $n$, $\mathbf{A}$ and $\mathbf{r}$ are the two-dimensional vectors in the plane of layers.

The term $F_{int}$ is the density of the interaction energy between the layers. One could write it in the following form

$$F_{int}(n) = \frac{\gamma_\perp \hbar^2}{d^2} |\Delta_{n+1} - \Delta_n|^2$$

where $\gamma_\perp$ is a constant that describes the strength of the interaction. For $d \to 0$, the sum is equivalent to

$$d\sum_n = \int dz$$

Therefore,

$$d\sum_n F_{int}(n) = \sum_n d\gamma_\perp \hbar^2 \left|\frac{\partial \Delta_n}{\partial z}\right|^2 = \gamma_\perp \hbar^2 \int dz \left|\frac{\partial \Delta_n}{\partial z}\right|^2$$

which gives the missing $z$-component of the gradient energy in Eq. (259).

However, Eq. (259) is not gauge invariant. To preserve the gauge invariance one needs to introduce the phase factors containing the vector potential

$$F_{int}(n) = \frac{\gamma_\perp \hbar^2}{d^2} \left|\Delta_{n+1} e^{-i\chi_A(n+1,n)} - \Delta_n\right|^2 \tag{260}$$

where

$$\chi_A(n+1,n) = \frac{2e}{\hbar c}\int_{z_n}^{z_{n+1}} A_z\, dz \tag{261}$$

For $d \to 0$, this expression will then go over into the correct expression

$$F_{int}(n) = \gamma_\perp \hbar^2 \left|\left(\frac{\partial \Delta}{\partial z} - \frac{2ie}{\hbar c} A_z\right)\right|^2$$

Finally, the free energy of the system of layers is

$$\begin{aligned}\mathcal{F}_{sn} = d\sum_n \int & \left[\alpha|\Delta_n(\mathbf{r}|^2 + \frac{\beta}{2}|\Delta_n(\mathbf{r}|^4 + \gamma_\| \left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Delta_n(\mathbf{r})\right|^2\right. \\ & \left. + \frac{\gamma_\perp \hbar^2}{d^2}\left|\Delta_{n+1}e^{-i\chi_A(n+1,n)} - \Delta_n\right|^2\right] d^2 r\end{aligned} \tag{262}$$

This energy is the basis for the Lawrence–Doniach model for layered superconductors.

Variation of the free energy with respect to the order parameter gives

$$\begin{aligned}\frac{\delta \mathcal{F}_{sn}}{\delta \Delta_n^*} = & \alpha\Delta_n + \beta|\Delta_n|^2\Delta_n - \gamma_\| \hbar^2 \left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)^2 \Delta_n(\mathbf{r}) \\ & -\frac{\gamma_\perp \hbar^2}{d^2}\left(\Delta_{n+1}e^{-i\chi_A(n+1,n)} + \Delta_{n-1}e^{+i\chi_A(n,n-1)} - 2\Delta_n\right) = 0\end{aligned} \tag{263}$$

The second line is the discrete analogue of the second derivative with respect to $z$.

Let us calculate the variation of the free energy with respect to $\delta A_z$. We have

$$\begin{aligned}\delta\mathcal{F}_{sn} &= -\frac{2ie\gamma_\perp\hbar}{dc}\sum_n \int d^2r\left[\Delta_{n+1}e^{-i\chi_A(n+1,n)}\left(\Delta_{n+1}^* e^{i\chi_A(n+1,n)} - \Delta_n^*\right) - c.c\right]\int_{z_n}^{z_{n+1}} \delta A_z\, dz \\ &= \frac{2ie\gamma_\perp\hbar}{dc}\sum_n \int d^2r\left[\Delta_{n+1}e^{-i\chi_A(n+1,n)}\Delta_n^* - \Delta_{n+1}^* e^{i\chi_A(n+1,n)}\Delta_n\right]\int_{z_n}^{z_{n+1}} \delta A_z\, dz\end{aligned}$$

Since

$$\int_{z_n}^{z_{n+1}} \delta A_z \, dz \approx \frac{d}{2} \left[ \delta A_z(n+1) + \delta A_z(n) \right]$$

we have

$$\delta \mathcal{F}_{sn} = \frac{ie\gamma_\perp \hbar}{dc} d \sum_n \int d^2 r \left[ \Delta_{n+1} e^{-i\chi_A(n+1,n)} \Delta_n^* - \Delta_{n+1}^* e^{i\chi_A(n+1,n)} \Delta_n \right] \\ \times \left[ \delta A_z(n+1) + \delta A_z(n) \right]$$

Changing the summation index $n$ for $n-1$ in the term with $A(n+1)$ we get

$$\delta \mathcal{F}_{sn} = \frac{ie\gamma_\perp \hbar}{dc} \, d \sum_n \int d^2 r \delta A_z(n) \left( \left[ \Delta_{n+1} e^{-i\chi_A(n+1,n)} \Delta_n^* - \Delta_{n+1}^* e^{i\chi_A(n+1,n)} \Delta_n \right] \right. \\ \left. + \left[ \Delta_n e^{-i\chi_A(n,n-1)} \Delta_{n-1}^* - \Delta_n^* e^{i\chi_A(n,n-1)} \Delta_{n-1} \right] \right)$$

Using

$$j_z = -c \frac{\delta \mathcal{F}}{\delta A_z}$$

we find

$$j_z = -\frac{ie\gamma_\perp \hbar}{d} \left( \left[ \Delta_{n+1} e^{-i\chi_A(n+1,n)} \Delta_n^* - \Delta_{n+1}^* e^{i\chi_A(n+1,n)} \Delta_n \right] \right. \\ \left. + \left[ \Delta_n e^{-i\chi_A(n,n-1)} \Delta_{n-1}^* - \Delta_n^* e^{i\chi_A(n,n-1)} \Delta_{n-1} \right] \right)$$

Putting

$$\Delta_n = |\Delta_n| e^{i\chi_n}$$

we finally obtain

$$j_z = \frac{2e\gamma_\perp \hbar}{d} \left[ |\Delta_{n+1} \Delta_n| \sin \left( \chi_{n+1} - \chi_n - \frac{2e}{\hbar c} \int_{z_n}^{z_{n+1}} A_z \, dz \right) \right. \\ \left. + |\Delta_n \Delta_{n-1}| \sin \left( \chi_n - \chi_{n-1} - \frac{2e}{\hbar c} \int_{z_{n-1}}^{z_n} A_z \, dz \right) \right] \tag{264}$$

### B. Anisotropic superconductors

If the distance between the layers is small $d \to 0$, one has for the free energy

$$\mathcal{F}_{sn} = \int \left[ \alpha|\Delta(\mathbf{r},z)|^2 + \frac{\beta}{2}|\Delta(\mathbf{r},z)|^4 + \gamma_{\|} \left| \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta(\mathbf{r},z) \right|^2 \right.$$
$$\left. + \gamma_{\perp} \left| \left( -i\hbar\frac{\partial \Delta}{\partial z} - \frac{2e}{c}A_z \right) \right|^2 \right] d^2r\, dz \qquad (265)$$

This expression is similar to Eq. (70); the distinction is that the constants $\gamma_{\|}$ and $\gamma_{\perp}$ are now different. These constants determine two coherence lengthes: one in the symmetry plane, $\xi_{\|}$, and another perpendicular to the plane, $\xi_{\perp}$,

$$\xi_{\|} = \hbar\sqrt{\gamma_{\|}/|\alpha|},\ \xi_{\perp} = \hbar\sqrt{\gamma_{\perp}/|\alpha|} \qquad (266)$$

Eq. (265) describes an anisotropic superconductor with uniaxial anisotropy.

If the order parameter is normalized in such a way that it is the wave function of superconducting electrons $\Psi$, the free energy has the form

$$\mathcal{F}_{sn} = \int \left[ a\,|\Psi|^2 + \frac{b}{2}\,|\Psi|^4 + \frac{1}{2m_{\|}} \left| \left( -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Psi \right|^2 \right.$$
$$\left. + \frac{1}{2m_{\perp}} \left| \left( -i\hbar\frac{\partial}{\partial z} - \frac{2e}{c}A_z \right) \Psi \right|^2 \right] dV. \qquad (267)$$

The constants $a$ and $b$ determine the magnetic field penetration length $\lambda_{\|}$

$$\frac{|a|}{b} = \frac{m_{\|}c^2}{16\pi e^2 \lambda_{\|}^2}$$

for currents flowing in the symmetry plane and also

$$\frac{|a|}{b} = \frac{m_{\perp}c^2}{16\pi e^2 \lambda_{\perp}^2}$$

for currents along the $z$ axis. The coherence lengthes along the plane and perpendicular to the plane are now also expressed through the effective masses

$$\xi_{\|}^2 = \frac{\hbar^2}{2m_{\|}|a|},\ \xi_{\perp}^2 = \frac{\hbar^2}{2m_{\perp}|a|}$$

### C. Upper critical field for parallel orientation

Let the magnetic field be parallel to the layers. We find a second-order phase transition into the superconducting state with lowering the field. We choose the $y$ axis along the field, with the gauge $\mathbf{A} = (0,\, 0,\, -Hx)$ and look for a solution in the form of

$$\Delta(n, \mathbf{r}) = Ce^{iqnd} f_q(x)$$

The equation for $f$ is obtained from Eq. (263). Using the definitions of $\xi_{||}$ and $\xi_{\perp}$ from Eq. (266) we arrive at the Mathieu equation:

$$\xi_{||}^2 \frac{\partial^2 f}{\partial x^2} - \frac{2\xi_{\perp}^2}{d^2}\left[1 - \cos\left(qd + \frac{2eHdx}{\hbar c}\right)\right] f + f = 0 \tag{268}$$

There are two limiting cases: (1) Weakly layered (nearly continuous) limit, and (2) Highly layered case.

#### *1. Continuous limit*

The continuous limit $\xi_{\perp} \gg d$ coincides with the anisotropic case. We shall see in a moment (see also Problem 2) that

$$H_{c2} = \frac{\Phi_0}{2\pi\xi_{||}\xi_{\perp}} \tag{269}$$

Since $x \sim \xi_{||}$ we find that the argument of the cosine function in Eq. (268) is

$$\frac{dx}{\xi_{||}\xi_{\perp}} \sim \frac{d}{\xi_{\perp}}$$

The continuous limit corresponds to

$$d \ll \xi_{\perp}$$

In this case, the cosine function in Eq. (268) can be expanded in small argument and the equation becomes

$$\xi_{||}^2 \frac{\partial f}{\partial x^2} - \xi_{\perp}^2 \left(q + \frac{2eHx}{\hbar c}\right)^2 f + f = 0$$

The lowest-level solution is

$$f = \exp\left[-\frac{1}{2\xi_{\|}^2}\left(x + \frac{\hbar c q}{2eH}\right)^2\right] \tag{270}$$

with $H = H_{c2}$ from Eq. (269).

### *2. Highly layered case*

As we shall see, this limit corresponds to a high upper critical field and is realized for a certain relation between $d$ and $\xi_\perp$. We introduce the parameter

$$s = \frac{2eHd^2\xi_{\|}}{\hbar c \xi_\perp} \tag{271}$$

For a continuous case,

$$s = \frac{d^2}{\xi_\perp^2} \ll 1$$

On the contrary, a highly layered case is realized for $s \gg 1$ when $H \to \infty$.

For this limit, a solution of Eq. (268) can be obtained by expansion in powers of a small $1/s$. The lowest-level periodic solution corresponds to an even Mathieu function

$$f = 1 + \frac{2}{s^2}\cos\left(\frac{2eHdx}{\hbar c}\right) + \frac{1}{2s^4}\cos\left(\frac{4eHdx}{\hbar c}\right) + \ldots$$

under the condition

$$s^2 = \frac{1}{1 - (d^2/2\xi_\perp^2)}$$

This condition determines the upper critical field

$$H_{c2} = \frac{\Phi_0}{2\pi d^2}\left(\frac{\xi_\perp}{\xi_{\|}}\right)\frac{1}{\sqrt{1 - (d^2/2\xi_\perp^2)}} \tag{272}$$

This solution is valid for $s \gg 1$, or $\xi_\perp(T) \to d/\sqrt{2}$.

In a situation such that $\xi_\perp(0) \sim d$, a weakly layered case with $\xi_\perp(T) \gg d$ at high temperatures can transform into a highly layered limit for lower temperatures. One can say that an anisotropic three-dimensional superconductor experiences a transition into effectively a two-dimensional superconductor at the temperature when $\xi_\perp(T) = d/\sqrt{2}$.

We observe that the upper critical field diverges as $\xi_\perp(T)$ approaches $d/\sqrt{2}$ with lowering the temperature. In this limit, a vortex cores fit in-between the superconducting layers, and the supercurrents do not destroy superconductivity.

Divergence of the upper critical field is similar to the Little and Parks effect for a non-singly-connected superconductor (Fig. 10) where the critical temperature is a periodic function of the magnetic field instead of vanishing with an increasing field. Suppose we have a cylinder of radius $R$ made of a thin film with thickness $d$ such that $d \ll \lambda$. The cylinder is placed in a magnetic field parallel to its axis. Because of the small thickness, the superconducting velocity inside the film is not fully screened:

$$\mathbf{v}_s = \frac{\hbar}{2m}\left(\nabla\chi - \frac{2e}{\hbar c}\mathbf{A}\right) \neq 0$$

Its circulation around the cylinder is

$$\int \mathbf{v}_s \, d\mathbf{l} = 2\pi R v_s = \frac{\hbar}{2m}\left(2\pi n - \frac{2e\Phi}{\hbar c}\right)$$

since the circulation of the phase can only be $2\pi n$ where $n$ is an integer. Here $\Phi$ is the flux inside the cylinder. Note that $\Phi$ is not quantized, because $d \ll \lambda_L$. Thus the superconducting velocity is

$$v_s = \frac{\hbar}{2mR}\left(n - \frac{\Phi}{\phi_0}\right)$$

We see that the superconducting velocity is a periodic function of the magnetic flux, the number of quanta $n$ being adjusted in such a way that $v_s$ is limited by

$$v_{s\,max} = \frac{\hbar}{4mR}$$

Superconductivity is not destroyed if this maximum value is smaller than the pair-breaking velocity $v_{pb} \sim \hbar/m\xi$. Instead, the critical temperature will oscillate: It is maximum when the flux is exactly integer number of flux quanta. The condition for the Little and Parks effect is thus $R \gg \xi$.

## Problems

### Problem 1

Derive Eq. (263)

### Problem 2

Find the upper critical field for an anisotropic superconductor in a magnetic field tilted by an angle $\theta$ with respect to the anisotropy axis (see Fig. 16).

# XI. JOSEPHSON JUNCTIONS. JOSEPHSON VORTICES.

## A. Josephson junctions

Assume that there are only two layers, $n = 1$ and $n = 2$, Fig. 30. The current between them will be

$$j_z = j_c \sin\left(\chi_2 - \chi_1 - \frac{2ed}{\hbar c} A_z\right) \tag{273}$$

where the quantity

$$j_c = \frac{4e\gamma_\perp \hbar}{d} |\Delta_1 \Delta_2|$$

is called the critical Josephson current. Equation (273) has the same form as the current through the short weak link.

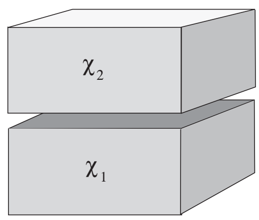


FIG. 30. The Josephson junction between two superconductors.

Equation (273) describes the d.c. Josephson effect: The supercurrent can flow through the insulating layer provided there is an interaction between the superconducting regions. In practice this interaction is realized via a quantum-mechanical tunneling of electrons through the insulating barrier. In this case, $\gamma_\perp \propto \exp(-Ap_F d/\hbar)$, where $A$ is a constant of order unity.

## B. Long Josephson junctions in the magnetic field.

Consider two large superconductors 1 and 2 separated by a thin insulating layer with a thickness $d$. The $(x, y)$ plane is in the middle of the insulating layer. The superconductor

1 is at $z < -d/2$, the superconductor 2 occupies the region $z > d/2$. The magnetic field is applied parallel to the insulating layer along the $x$ axis, Fig. 31. We choose $\mathbf{A} = (0,\, A_y,\, 0)$. Therefore,

$$h_x = -\frac{\partial A_y}{\partial z}$$

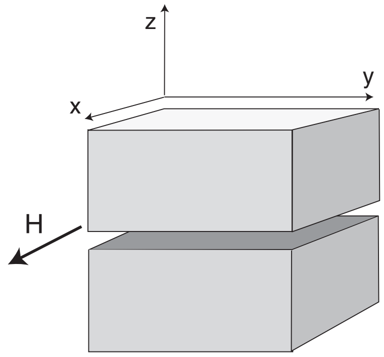


FIG. 31. A long Josephson junction in a magnetic field.

Deep in the superconductors

$$\frac{\partial \chi}{\partial y} = \frac{2e}{\hbar c} A_y$$

due to the Meissner screening. The field decays as

$$h_x(z) = \begin{cases} H e^{-(z-d/2)/\lambda_2} & z > d/2 \\ H e^{(z+d/2)/\lambda_1} & z < -d/2 \end{cases}$$

We denote $H$ the field in between the two superconductors at $z = 0$; $\lambda_{1,2}$ is the London penetration length in the superconductor 1 or 2, respectively. The phase $\chi$ is essentially independent of $z$ because the current through the junction is small. In the bulk it is thus the same as at the boundary of the contact.

The vector potential deep in the superconductor 2 is

$$A_y(2) = -\int_0^\infty h_x\, dz = -\int_0^{d/2} h_x\, dz - \int_{d/2}^\infty h_x\, dz = -H\frac{d}{2} - H\lambda_2$$

Here we put $A_y(0) = 0$ and assume that $H$ is independent of $z$ at $0 < z < d/2$. Similarly,

$$
A_y(1) = -\int_0^{-\infty} h_x\, dz = \int_{-d/2}^{0} h_x\, dz + \int_{-\infty}^{d/2} h_x\, dz = H\left(\frac{d}{2} + \lambda_1\right)
$$

As a result

$$
H = \frac{\hbar c}{2e(\lambda_1 + d/2)} \frac{\partial \chi_1}{\partial y} = -\frac{\hbar c}{2e(\lambda_2 + d/2)} \frac{\partial \chi_2}{\partial y}
$$

so that

$$
\frac{\partial \varphi}{\partial y} = -\frac{2e(\lambda_1 + \lambda_2 + d)H}{\hbar c} \tag{274}
$$

where

$$
\varphi = \chi_2 - \chi_1
$$

The magnetic field $H$ is generally a function of the coordinate $y$ into the junction.

Using the Maxwell equation

$$
\frac{c}{4\pi} \mathrm{curl}_z\, \mathbf{h} = j_c \sin\varphi \quad \text{or} \quad -\frac{c}{4\pi}\frac{\partial H}{\partial y} = j_c \sin\varphi
$$

we obtain

$$
\frac{\hbar c^2}{8\pi e(\lambda_2 + \lambda_1 + d)} \frac{\partial^2 \varphi}{\partial y^2} = j_c \sin\varphi
$$

or

$$
\lambda_J^2 \frac{\partial^2 \varphi}{\partial y^2} = \sin\varphi \tag{275}
$$

where

$$
\lambda_J = \sqrt{\frac{\hbar c^2}{8\pi e j_c(\lambda_2 + \lambda_1 + d)}} \tag{276}
$$

is called the Josephson length. We can write Eq. (274) in the form

$$
H = -\frac{4\pi j_c \lambda_J^2}{c} \frac{\partial \varphi}{\partial y} \tag{277}
$$

Equation (275) is similar to equation of motion for a pendulum. Indeed, the latter has the form

$$\frac{d\theta^2}{dt^2} = -\frac{g}{l}\sin\theta$$

where the angle $\theta$ is measured from the bottom. Equation (275) is obtained from it by replacing $\theta = \pi - \varphi$ and putting $\lambda_J^{-2} = g/l$. This means that the pendulum angle $\phi$ is measured from the top, Fig. 32.

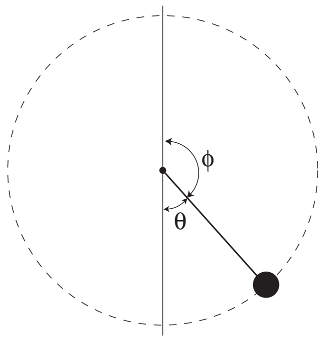


FIG. 32. A pendulum moves in time similar to variations of $\phi$ along the $y$ axis.

Consider first a low field $H$ applied outside the junction at $y = 0$. In this case $\varphi$ is also small. Expanding Eq. (275) we find

$$\lambda_J^2 \frac{\partial^2 \varphi}{\partial y^2} = \varphi$$

whence (see Fig. 33, curve 1)

$$\varphi = \varphi_0 e^{-y/\lambda_J}$$

The magnetic field decays as

$$H(y) = H(0)e^{-y/\lambda_J}$$

where we find from Eq. (277)

$$H(0) = \frac{4\pi j_c \lambda_J \varphi_0}{c} \tag{278}$$

Magnetic field decays into the junction in a way similar to the Meissner effect. The penetration length is $\lambda_J$. This length is larger than $\lambda_L$ because the screening current cannot exceed the Josephson critical current $j_c$. Such behavior exists for fields smaller than $4\pi j_c \lambda_J / c$.

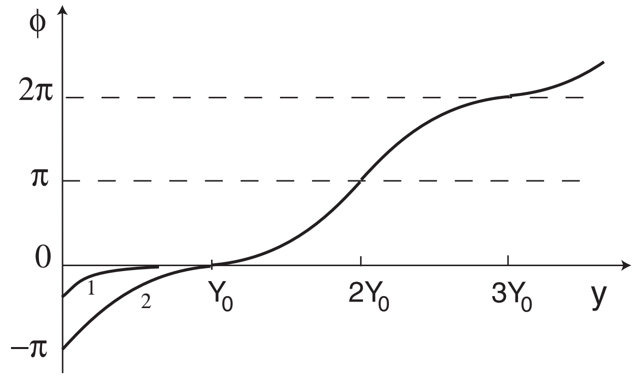


FIG. 33. The phase difference $\varphi$ as a function of the distance into the junction measured from the left edge. Curve 1: small magnetic fields. Curve 2: large fields, phase runs from $\varphi = -\pi$ at the edge through $2\pi n$ values making Josephson vortices. The curves correspond to $H(0) < 0$.

To find a solution of Eq. (275) for larger fields, we multiply it by $\partial\varphi/\partial y$ and obtain

$$\frac{\lambda_J^2}{2}\left(\frac{\partial\varphi}{\partial y}\right)^2 + \cos\varphi = C \tag{279}$$

where $C$ is a constant. Eq. (279) gives

$$\frac{\lambda_J}{\sqrt{2}}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\sqrt{C-\cos\varphi}} = y \tag{280}$$

As discussed above, the stable solution for $y \to \infty$ corresponds to $\varphi = 0$ and $\partial\varphi/\partial y = 0$. Eq. (279) results in $C = 1$. The applied filed is then

$$H(0) = -\frac{4\pi j_c \lambda_j^2}{c}\frac{\partial\varphi(0)}{\partial y} = \frac{8\pi j_c \lambda_j}{c}\sin\frac{\varphi_0}{2}$$

The sign is chosen to agree with Eq. (278) for small fields. Increasing field leads to an increase in $\varphi_0$ until it reaches $\pm\pi$. This threshold corresponds to the filed

$$H_1 = \frac{8\pi j_c \lambda_J}{c}$$

Above this field, the constant $C > 1$, and the phase $\varphi$ can vary within unlimited range. Consider for example the case $H(0) < 0$. The phase runs indefinitely from $-\pi$ at the edge through the values $2\pi n$ producing the so called solitons (Fig. 33, curve 2). The phase solitons are also called the Josephson vortices: the phase difference across the junction varies by $2\pi$ each time as we go past one Josephson vortex. The distance between vortices is $L = 2Y_0 \sim \lambda_J$.

## Problems

### Problem 1

Find the distance between Josephson vortices for $H$ close to $H_1$.

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