Tfy-X Quantum computing 2. Exercise Spring 2007 26.3.2007
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1. Rabi oscillation.
Let us consider a Hamiltonian

$$H = \hbar\omega\sigma_2 = \hbar\omega \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1}$$

Suppose the system is in the initial state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

(1) Find the wave function $|\psi(t)\rangle$ at time t.
(2) Find the probability of the system having eigenvalue $\sigma_z = +1$ at $t > 0$.
(3) Find the probability of the system having eigenvalue $\sigma_x = +1$ at $t > 0$.

2. Entanglement.
a) Which of the following states are entangled states? If the state is a tensor product, decompose it to the product form.

(1) $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$
(2) $\frac{1}{\sqrt{3}}(|000\rangle + |100\rangle + |101\rangle)$
(3) $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
(4) $\frac{1}{\sqrt{2}}(|101\rangle + |111\rangle)$
(5) $\frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle)$
(6) $\frac{1}{2}(|000\rangle + |010\rangle + |011\rangle + |111\rangle)$
(7) $\frac{1}{2}(|000\rangle + |001\rangle + |100\rangle + |101\rangle)$

b) Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Find the expectation value of $\sigma_x \otimes \sigma_z$ measured in this state.

3. Two-state system and "environment".
Consider a two-state quantum system with an environment given by a third state. Consider a Hamiltonian

$$H = \hbar\omega \begin{pmatrix} 0 & 1 & \epsilon\sqrt{2} \\ 1 & 0 & \epsilon\sqrt{2} \\ \epsilon\sqrt{2} & \epsilon\sqrt{2} & 1 \end{pmatrix}, \tag{3}$$

where $\epsilon \in \mathbb{R}$. Assume that the system is in the initial state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

(1) Find the wave function $|\psi(t)\rangle$ at time t.
(2) How much "information" has leaked into the third state at time $t > 0$, i.e., what is the probability that upon observation, the system collapses into the state

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}? \tag{5}$$

4. Density matrix.
Verify that

$$\rho = \begin{pmatrix} \frac{p}{2} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\ 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{p}{2} \end{pmatrix}, \qquad (0 \leq p \leq 1) \tag{6}$$

is a density matrix. Show that the negativity vanishes only for $p = 1/2$.