Exercise 1

(On Mon Sep 26th at 12:15, in F2. Participants should present their solutions on the blackboard.)

1.1 The charging energy required to bring a single electron between two capacitor plates is $E_C = e^2/(2C)$. Let us assume a simple parallel-plate model for a capacitor, $C = \epsilon A/d$, where $\epsilon = \epsilon_r \epsilon_0$, typical $\epsilon_r = 10$ and $\epsilon_0 = 8.85$ pF/m, A is the area of the plates and d their separation. We can also set $d = 1$ nm, a typical thickness of the oxide layer ("tunnel contact") formed between two metals. Assuming a square plate, $A = w^2$, estimate a width w of the junction which would correspond to the charging energy E_C being equal to 1 K. Estimate also the corresponding capacitances. How should these scales change so that charging effects would be observable at room temperature, i.e., $E_C \approx 300K/k_B$? Remember that $e = 1.6\ 10^{-19}$ C and $k_B = 1.38\ 10^{-23}$ J/K.

1.2 Show that the resistance of a tunnel barrier is independent of temperature or voltage.

1.3 Show that the heat current through a tunnel barrier in the linear response regime obeys a Wiedemann-Franz law, i.e., $\dot{Q}/\Delta T \propto T/R_T$. Find also the prefactor of this expression. *Hint: Assume a vanishing voltage and that the temperatures $T_{L/R}$ of the left/right reservoirs are $T_{L/R} = T \pm \Delta T/2$. Finally, take the linear order in ΔT.*

1.4 Prove Eqs. (1.19) and (1.21),

$$a_i^\dagger a_i = N_i, \quad a_i a_i^\dagger = 1 - N_i, \tag{1.19}$$

$$\{a_i, a_j^\dagger\} \equiv a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij} \tag{1.21a}$$

$$a_i a_j + a_j a_i = a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0, \tag{1.21b}$$

by using the definitions (1.17) and (1.18) of the annihilation operator a_i.

1.5 The Hamiltonian for a Harmonic oscillator is

$$H = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}),$$

where $\omega = \sqrt{k/m}$ is the resonance frequency for a mass m and spring constant k characterizing the potential, and $\hat{a}$ is the bosonic annihilation operator. With these operators, the (Heisenberg) operators for position $\hat{x}$ and and momentum $\hat{p}$ are $\hat{x} = \sqrt{\hbar/(2m\omega)}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = i\sqrt{\hbar m\omega/2}(\hat{a}^\dagger - \hat{a})$. From the Heisenberg equation of motion,

$$i\hbar\frac{d\hat{O}}{dt} = [\hat{O}, H],$$

valid for any operator $\hat{O}$ without external time dependence, derive the equations of motion for $\hat{x}$ and $\hat{p}$.