Hydrodynamics of superfluid liquid

I. M. Khalatnikov
Landau Institute for Theoretical Physics

The relativistic hydrodynamics of superfluid liquid :

- nonrelativistic equation for pressure P

$$dP = \rho d\mu + sdT + \boldsymbol{g} d\boldsymbol{v}_n - (\boldsymbol{j} - \rho \boldsymbol{v}_n) d\boldsymbol{v}_n$$

- in relativistic case, density of flow $\boldsymbol{j}$ does not coincide with density of momentum $\boldsymbol{g}$. The definition of $\boldsymbol{w}$

$$\boldsymbol{g} = \rho \boldsymbol{v}_s + s\boldsymbol{w}$$

- introduce four-vectors

$$v_\mu = (\mu + \boldsymbol{v}_s \boldsymbol{v}_n - \boldsymbol{v}_s), \quad w_\mu = (T + \boldsymbol{v}_n \boldsymbol{w} - \boldsymbol{w})$$

- for pressure

$$dP = j^\mu dv_\mu + s^\mu dw_\mu$$

where $j^\mu = (\rho, \boldsymbol{j})$ - density of 4-vector of flow of mass, $s^\mu = (s, s\boldsymbol{v}_n)$ - density of 4-vector of flow of entropy

Variational principle:

- pressure P coincides with the Lagrangian density L

- we introduce Clebsch-variables

$$v_\mu = -\nabla_\mu \alpha$$

$$w_\mu = -\nabla_\mu \xi - \varphi \nabla_\mu \gamma$$

- the variation of action A

$$\int d^4x(-g)^{1/2} P(v_\mu, w_\mu)$$

gives us equations

$$\nabla_\mu j^\mu = 0$$

$$\nabla_\mu s^\mu = 0$$

$$\nabla_\mu v_\nu - \nabla_\nu v_\mu = 0$$

$$s^\mu(\nabla_\mu w_\nu - \nabla_\nu w_\mu) = 0$$

Quantum vortices:

- the contour integral around a vortex

$$\oint dx^{\mu} v_{\mu} = 2\pi\hbar/m$$

- the number of vortices N crossing the surface

$$N = \frac{m}{2\pi\hbar} \int df^{\mu\nu} (\nabla_{\mu} v_{\nu} - \nabla_{\nu} v_{\mu})$$

- energy-momentum tensor

$$T^{\mu}_{\nu} = \frac{\partial P}{\partial v_{\mu}} v_{\nu} + 2 \frac{\partial P}{\partial(\nabla_{\mu} v_{\lambda} - \nabla_{\lambda} v_{\mu})} (\nabla_{\nu} v_{\lambda} - \nabla_{\lambda} v_{\nu}) - \delta^{\mu}_{\nu} P$$

- conservation of energy and momentum

$$\nabla_\mu T^\mu_\mu = 0$$

gives us two equations

$$j^\mu(\nabla_\mu v_\nu - \nabla_\nu v_\mu) = 0$$

$$\nabla_\mu j^\mu = 0$$

Equation of hydrodynamics and symmetry group

- the generators of the group are denoted as $\widehat{G}_a$ and the structural constant is τ^c_{ab}. Then the Poisson bracket defines the group of symmetry

$$\widehat{G}_a\widehat{G}_b - \widehat{G}_b\widehat{G}_a = i\tau^c_{ab}\widehat{G}_c$$

- variation of order parameter ψ

$$\delta\psi = i\delta\alpha^a\widehat{G}_a\psi$$

where $\delta\alpha^a$ are variations of the “rotation angles”

- for the gradient

$$\nabla\psi = i\boldsymbol{W}^a\widehat{G}_a\psi$$

- when the curl of the right hand side is equal to zero

$$\omega_{ik}^a = 0$$

where

$$\omega_{ik}^a = \nabla_i W_k^a - \nabla_k W_i^a + \tau_{bc}^a W_i^b W_k^c, \quad \omega_{ik}^a = \frac{1}{2}\epsilon_{ikn}\omega_n^a$$

- $\boldsymbol{W}^a$ are chiral velocities
- momentum density $\boldsymbol{g}$

$$\boldsymbol{g} = G_a \boldsymbol{W}^a$$

where G_a are the densities of generators

- for nonzero temperature we have to add the momentum density $\boldsymbol{p}$ of the normal part

$$\boldsymbol{g} = G_a \boldsymbol{W}^a + \boldsymbol{p}$$

- Poisson brackets for densities G_a and $\boldsymbol{W}^a$

$$\{G_a(\boldsymbol{r}_1), G_b(\boldsymbol{r}_2)\} = -\tau^c_{ab} G_c \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

$$\{G_a(\boldsymbol{r}_1), W^b_i(\boldsymbol{r}_2)\} = \delta^b_a \nabla_i \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \tau^b_{ac} W^c_i \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

- in the presence of singular solitons, when $\omega^a \neq 0$ we assume

$$\{G_a(\boldsymbol{r}_1), \omega^b(\boldsymbol{r}_2)\} = \tau^b_{ac} \omega^c \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

- we assume that in $\{W, W\}$ there are no derivatives of δ-function:

$$\{W^a_i(\boldsymbol{r}_1), W^b_k(\boldsymbol{r}_2)\} = \phi^{ab}_{ik} \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

- here

$$\phi_{ik}^{ab} = F_c^{ab}\omega_{ik}^c$$

- non-dissipative equations of hydrodynamics constructed from Hamiltonian

$$H = \int d^3 r E$$

where density E is defined by

$$dE = \mu^a dG_a + \boldsymbol{\eta}_a d\boldsymbol{W}^a + \boldsymbol{\lambda}_a d\boldsymbol{\omega}^a + \boldsymbol{V}_n d\boldsymbol{p} + T d\zeta$$

- equations of motion can be obtained from Poisson brackets

  and

$$
\begin{aligned}
\frac{\partial G_a}{\partial t} &= \{H, G_a\} = -\nabla \boldsymbol{\eta}_a + (\mu^b \tau^c_{ab} G_c - \boldsymbol{\lambda}_b \tau^b_{ac} \omega^c - \boldsymbol{W}^c \tau^b_{ac} \boldsymbol{\eta}_b) \\
\frac{\partial \zeta}{\partial t} &= \{H, \zeta\} = -\nabla(\zeta \boldsymbol{V}_n) \\
\frac{\partial \boldsymbol{W}^a}{\partial t} &= \{H, \boldsymbol{W}^a\} \\
\frac{\partial g_i}{\partial t} &= \{H, g_i\}
\end{aligned}
$$

Superfluid ^{4}He

- in the presence of vortices, the curl of the average superfluid velocity is not equal to zero

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{W}$$

- if we assume

$$F = -1/\rho_s$$

then we find the Poisson bracket

$$\{W_i(\boldsymbol{r}_1), W_j(\boldsymbol{r}_2)\} = -1/\rho_s \omega_{ij} \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

- equations of superfluid hydrodynamics

$$\frac{\partial \rho}{\partial t} = -\nabla \boldsymbol{g}, \qquad \boldsymbol{g} = G_a \boldsymbol{W}^a + \boldsymbol{p}$$

$$\frac{\partial \boldsymbol{W}}{\partial t} = -\nabla \mu + \frac{1}{\rho_s}(\boldsymbol{g} - \rho_n \boldsymbol{V}_n + \nabla \times \boldsymbol{\lambda}) \times \boldsymbol{\omega}$$

Lagrangian method of superfluid hydrodynamics

- the Lagrangian is

$$L = \frac{1}{2}\rho \boldsymbol{v}_s^2 + \boldsymbol{p}\boldsymbol{v}_s - \tilde{\epsilon}(\rho, s, \boldsymbol{v}_n - \boldsymbol{v}_s) +$$
$$+ \alpha(\dot{\rho} + \nabla \boldsymbol{j}) + \beta(\dot{s} + \nabla(s\boldsymbol{v}_n)) + \gamma(\dot{f} + \nabla(f\boldsymbol{v}_n))$$

where $\tilde{\epsilon} = \epsilon - \boldsymbol{p}(\boldsymbol{v}_n - \boldsymbol{v}_s)$ and

$$\boldsymbol{j} = \rho \boldsymbol{v}_s + \boldsymbol{p}$$

$$d\epsilon = T ds + \mu d\rho + (\boldsymbol{v}_n - \boldsymbol{v}_s, d\boldsymbol{p})$$

Lagrangian method of superfluid hydrodynamics

- variation of the Lagrangian gives

$$
\begin{aligned}
\boldsymbol{j}: &\quad \boldsymbol{v}_s = \nabla\alpha \\
\boldsymbol{v}_n: &\quad \boldsymbol{p} = s\nabla\beta + f\nabla\gamma \\
\boldsymbol{v}_s: &\quad \boldsymbol{j} = \rho\boldsymbol{v}_s + \boldsymbol{p} = \rho\nabla\alpha + s\nabla\beta + f\nabla\gamma \\
\rho: &\quad \dot{\alpha} + \mu + \frac{\boldsymbol{v}_s^2}{2} = 0 \\
s, f: &\quad \dot{\beta} - \boldsymbol{v}_n\nabla\beta - T = 0, \quad \dot{\gamma} + \boldsymbol{v}_n\nabla\gamma = 0,
\end{aligned}
$$

Hamiltonian formulation

- definition of momentum

$$\boldsymbol{j} = \sum p \nabla q$$

momentum	coordinate
ρ	α
s	β
f	γ

- Hamiltonian equations

$$\dot{\alpha} = \frac{\delta H}{\delta p}, \qquad \dot{\rho} = -\frac{\delta H}{\delta \alpha}$$
$$\dot{\beta} = \frac{\delta H}{\delta s}, \qquad \dot{s} = -\frac{\delta H}{\delta \beta}$$
$$\dot{\gamma} = \frac{\delta H}{\delta f}, \qquad \dot{f} = -\frac{\delta H}{\delta \gamma}$$

- the Hamiltonian

$$H = \int d^3 x E$$