Simple Models of Complex Turbulent Flows

V.S. L'vov (VL)

The Weizmann Institute of Science (WIS)

Outline

1. One-fluid model for a turbulently flowing suspension

   VL, G. Ooms (Burgescentrum, The Netherlands), A. Pomyalov, (WIS)

2. Model of clustering of inertial particles in a turbulent flows

   VL, T. Elperin, N. Kleeorin, I. Rogachevskii (Ben-Gurion Univ. of Negev), D. Sokoloff (Moscow Univ.), M. Liberman (Uppsala Univ, Sweden)

3. Analytical model of the turbulent boundary layer

   VL, A. Pomyalov, V. Tiberkevich (WIS)

Preliminary:

• The Euler and Navier-Stokes equations

The Euler equation for $\boldsymbol{v}(\boldsymbol{r},t)$ is the 2nd Newton's law for the fluid particle:

$$\rho\Big[\overbrace{\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}}^{\text{Fluid particle Acceleration}}\Big] \quad - \quad \overset{\text{Pressure Force}}{(-\boldsymbol{\nabla}p)} \quad = 0\,, \qquad \text{Leonard Euler, 1741.}$$

The Navier-Stokes equation accounts for the viscous friction:

$$\rho\Big[\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t}+\underbrace{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\Big]+\boldsymbol{\nabla}p}_{\text{Nonlinear interaction}} = \underbrace{(\rho\,\nu)\,\Delta\boldsymbol{v}}_{\text{viscous friction}}\,, \quad \text{Claude L.M.H. Navier, 1827,}$$

George Gabriel Stokes, 1845.

Osborne Reynolds (1894) introduced "Reynolds number" $\mathcal{R}e$

$$\mathcal{R}e = \frac{\blacksquare}{\blacksquare} \simeq \frac{u\nabla v}{\nu\Delta v} \simeq \frac{LV}{\nu}$$ as a measure of the nonlinearity of the NSE.

- **Lewis Fry Richardson (1920) cascade model of turbulence:**

"Big whirls have little whirls

That feed on their velocity

And little whirls have lesser whirls

And so on to viscosity"

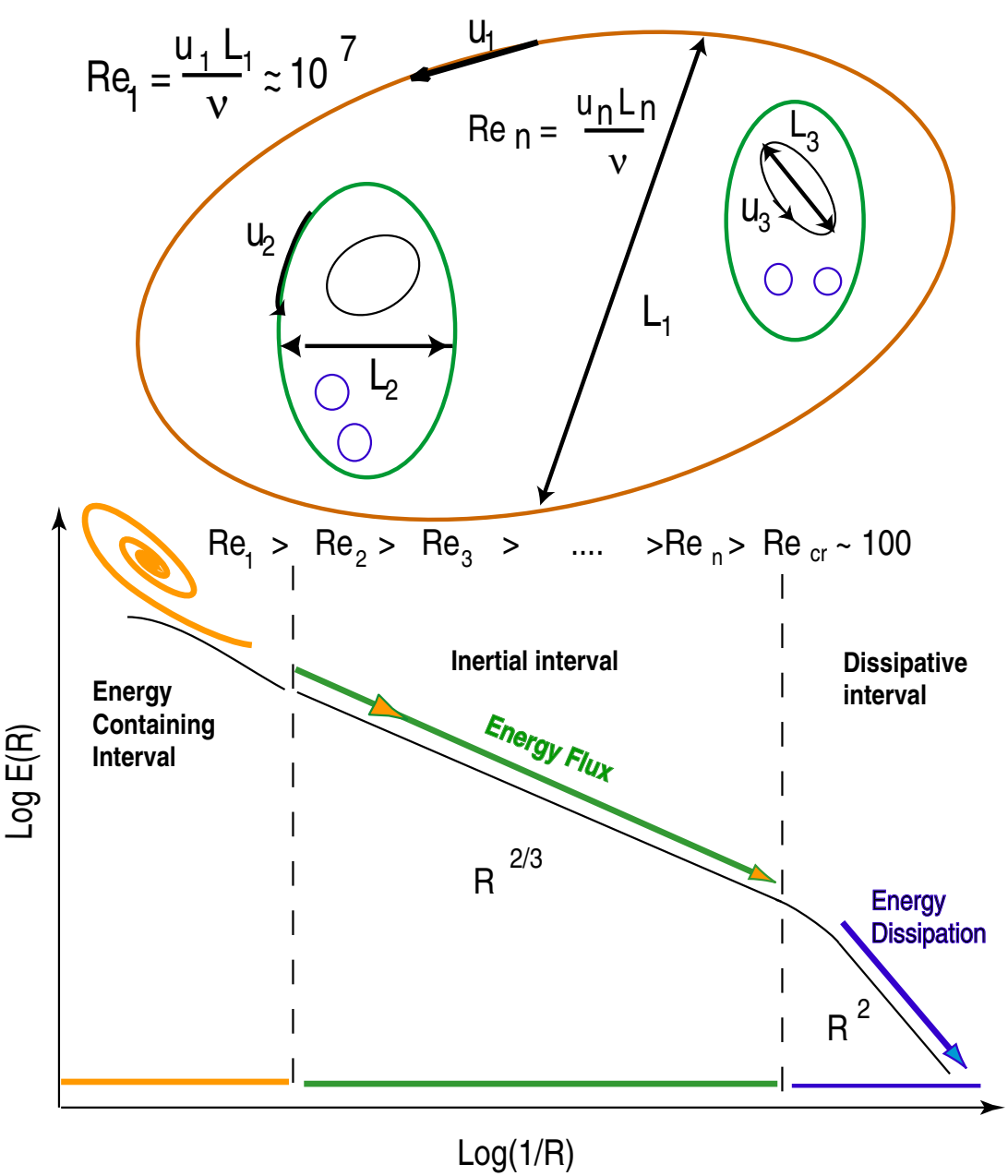
L.F. Richardson, paraphrase of J. Swift

$\Leftarrow$ Hurricane Bonnie, $V_{\mathsf{T}} \simeq 300\frac{\mathsf{m}}{\mathsf{s}}$,
Reynolds number at $H \simeq 500\mathsf{m}$

$$\mathcal{R}e = \frac{V_{\mathsf{T}} H}{\nu} \simeq 10^{10} \gg \mathcal{R}e_{\mathsf{cr}} \sim 10^2$$

Unstable H, V_{T}-eddies create smaller H_1, V_1-eddies with $\mathcal{R}e > \mathcal{R}e_1 \gg \mathcal{R}e_{\mathsf{cr}}$. Their instability creates H_2, V_2-eddies of the second generation, end so on, until $\mathcal{R}e_n$ of the n-th generation eddies reaches R_{cr} and will be dissipated by viscosity: $\mathcal{R}e > \mathcal{R}e_1 > \mathcal{R}e_2 > \ \ldots \ > \mathcal{R}e_{n-1} > \mathcal{R}e_n > \mathcal{R}e_{\mathsf{cr}}$.

Andrei N. **Kolmogorov-1941 cascade model of homogeneous turbulence:**



I. Universality of small scale statistics, isotropy, homogeneity;

II. Scale-by-scale "locality" of the energy transfer;

III. In the inertial interval of scales the only relevant parameter is the mean energy flux ε .

$\Rightarrow$ dimensional reasoning $\Rightarrow$

1. Turbulent energy of scale ℓ in inertial interval $E_\ell \simeq \rho\, \varepsilon^{2/3}\, \ell^{2/3}$,
2. Turnover and life time of ℓ-eddies: $\tau_\ell \simeq \varepsilon^{-1/3}\, \ell^{2/3}$
3. Viscous crossover scale $\eta \simeq \varepsilon^{-1/4}\, \nu^{3/4}$, $N \sim \mathcal{R}e^{3/4} \ldots$

1. One-fluid model for a turbulently flowing suspension

VL, G. Ooms A. Pomyalov [PRE, **67**, 046314, (2003)]

• Basic idea of One-fluid Approximation: Statistical ensemble of all particles, (partially involved in the motion of k-eddies) $\Rightarrow$ two sub-ensembles of particles: "fully comoving" fraction $f_{\rm com}(k)$ & "fully resting" fraction $f_{\rm rest}(k)$

$$f_{\rm rest}(k) = 1 - f_{\rm com}(k) = \frac{[\tau_{\rm p}\gamma(k)]^2}{[1+\tau_{\rm p}\gamma(k)]^2}\,,$$

$\tau_{\rm p}$ – particle response time,

$\gamma(k)$ – turnover frequency of k-eddies

a. Effective Density of Suspensions (ϕ – mass fraction) :

$$\rho_{\rm eff}(k) = \rho_{\rm f}[1+\phi f_{\rm com}(k)] = \rho_{\rm f}\Big\{1+\phi\,\frac{1+2\,\tau_{\rm p}\gamma(k)}{[1+\tau_{\rm p}\gamma(k)]^2}\Big\}\,.$$

• Basic idea of One-fluid Approximation: Statistical ensemble of all particles,

(partially involved in the motion of k-eddies) $\Rightarrow$ two sub-ensembles of particles:

"fully comoving" fraction $f_{\mathrm{com}}(k)$ & "fully resting" fraction $f_{\mathrm{rest}}(k)$

$$f_{\mathrm{rest}}(k) = 1 - f_{\mathrm{com}}(k) = \frac{[\tau_{\mathrm{p}}\gamma(k)]^2}{[1+\tau_{\mathrm{p}}\gamma(k)]^2},$$

τ_{p} – particle response time,

$\gamma(k)$ – turnover frequency of k-eddies

a. *Effective Density of Suspensions* (ϕ – mass fraction) :

$$\rho_{\mathrm{eff}}(k) = \rho_{\mathrm{f}}[1+\phi f_{\mathrm{com}}(k)] = \rho_{\mathrm{f}}\Big\{1+\phi\,\frac{1+2\,\tau_{\mathrm{p}}\gamma(k)}{[1+\tau_{\mathrm{p}}\gamma(k)]^2}\Big\}\,.$$

b. *Fluid-particle friction*

$$\gamma_{\mathrm{p}}(k) = \frac{M_{\mathrm{p}}}{\tau_{\mathrm{p}}\,M_{\mathrm{eff}}} = \frac{\phi\,\rho_{\mathrm{f}}\,f_{\mathrm{rest}}(k)}{\tau_{\mathrm{p}}\,\rho_{\mathrm{eff}}(k)} = \frac{\phi\,\tau_{\mathrm{p}}\gamma^2(k)}{(1+\phi)[1+2\tau_{\mathrm{p}}\gamma(k)]+\tau_{\mathrm{p}}^2\gamma^2(k)}\,.$$

• Basic idea of One-fluid Approximation: Statistical ensemble of all particles,

(partially involved in the motion of k-eddies) $\Rightarrow$ two sub-ensembles of particles:

"fully comoving" fraction $f_{\rm com}(k)$ & "fully resting" fraction $f_{\rm rest}(k)$

$$f_{\rm rest}(k) = 1 - f_{\rm com}(k) = \frac{[\tau_{\rm p}\gamma(k)]^2}{[1+\tau_{\rm p}\gamma(k)]^2}\,,$$

$\tau_{\rm p}$ – particle response time,

$\gamma(k)$ – turnover frequency of k-eddies

a. *Effective Density of Suspensions* (ϕ – mass fraction) :

$$\rho_{\rm eff}(k) = \rho_{\rm f}[1+\phi f_{\rm com}(k)] = \rho_{\rm f}\Big\{1+\phi\,\frac{1+2\,\tau_{\rm p}\gamma(k)}{[1+\tau_{\rm p}\gamma(k)]^2}\Big\}\,.$$

b. *Fluid-particle friction*

$$\gamma_{\rm p}(k) = \frac{M_{\rm p}}{\tau_{\rm p}\,M_{\rm eff}} = \frac{\phi\,\rho_{\rm f}\,f_{\rm rest}(k)}{\tau_{\rm p}\,\rho_{\rm eff}(k)} = \frac{\phi\,\tau_{\rm p}\gamma^2(k)}{(1+\phi)[1+2\tau_{\rm p}\gamma(k)]+\tau_{\rm p}^2\gamma^2(k)}\,.$$

c. *Effective interaction amplitude* in the modified Navier-Stokes Eq.

is Galilean invariant and conserves the total kinetic energy

• Budget of the Kinetic Energy in Suspensions

Exact Energy Budget Equation: $$\frac{\partial \mathcal{E}(t,k)}{2\,\partial t} + \gamma_{\mathsf{p}}(k)\mathcal{E}(t,k) + \frac{d\varepsilon(k)}{dk} = 0\ .$$

– Energy spectrum of suspension: $\mathcal{E}(t,k) = \rho_{\mathsf{eff}}(k)\, k^2 \left\langle |\boldsymbol{v}_k(t)|^2 \right\rangle / 2\pi\,,$

– $\varepsilon(k)$ - energy flux on the scale $\boldsymbol{k}$, that is $\neq$ rate of energy dissipation, due to the fluid-particle friction, $\gamma_{\mathsf{p}}(k)$.

• Budget of the Kinetic Energy in Suspensions

Exact Energy Budget Equation: $$\frac{\partial \mathcal{E}(t,k)}{2\,\partial t} + \gamma_{\rm p}(k)\mathcal{E}(t,k) + \frac{d\varepsilon(k)}{dk} = 0\ .$$

– Energy spectrum of suspension: $\mathcal{E}(t,k) = \rho_{\rm eff}(k)\, k^2 \left\langle |\boldsymbol{v}_k(t)|^2 \right\rangle / 2\pi\,,$

– $\varepsilon(k)$ - energy flux on the scale $\boldsymbol{k}$, that is $\neq$ rate of energy dissipation, due to the fluid-particle friction, $\gamma_{\rm p}(k)$.

• Richardson-Kolmogorov Closure for $\mathcal{E}(k)$ & $\gamma(k)$, turnover frequency:

$$\mathcal{E}(k) = C_1\,[\varepsilon(k)^2 \rho_{\rm eff}(k)]^{1/3} k^{-5/3}\,, \qquad \gamma(k) = C_2\,[\varepsilon(k)/\rho_{\rm eff}(k)]^{1/3} k^{2/3} \quad \Rightarrow$$

Energy flux Eq.: $$\frac{d\varepsilon(k)}{dk} + \frac{\varepsilon(k)}{k}\,\frac{C_1 C_2\,\phi\,\gamma(k)\tau_{\rm p}}{(1+\phi)[1+2\,\gamma(k)\tau_{\rm p}] + [\gamma(k)\tau_{\rm p}]^2} = 0 \Rightarrow$$

• 3 Budget of the Kinetic Energy in Suspensions

Exact Energy Budget Equation: $$\frac{\partial \mathcal{E}(t,k)}{2\,\partial t} + \gamma_{\rm p}(k)\mathcal{E}(t,k) + \frac{d\varepsilon(k)}{dk} = 0\ .$$

– Energy spectrum of suspension: $\mathcal{E}(t,k) = \rho_{\rm eff}(k)\,k^2\left\langle |\boldsymbol{v}_k(t)|^2\right\rangle/2\pi\,,$

– $\varepsilon(k)$ - energy flux on the scale $\boldsymbol{k}$, that is $\neq$ rate of energy dissipation, due to the fluid-particle friction, $\gamma_{\rm p}(k)$.

• Richardson-Kolmogorov Closure for $\mathcal{E}(k)$ & $\gamma(k)$, turnover frequency:

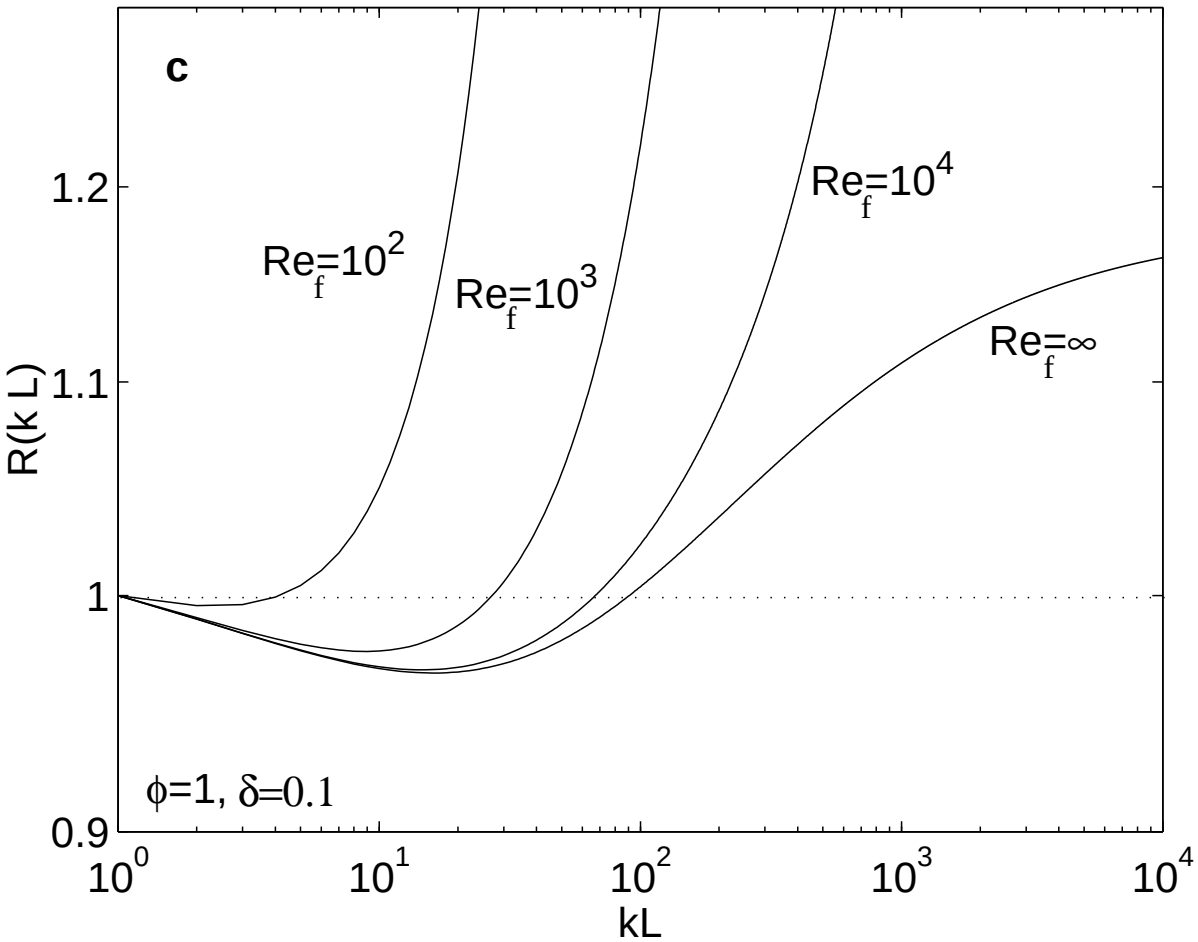
$$\mathcal{E}(k) = C_1\,[\varepsilon(k)^2\rho_{\rm eff}(k)]^{1/3}k^{-5/3}\,, \qquad \gamma(k) = C_2\,[\varepsilon(k)/\rho_{\rm eff}(k)]^{1/3}k^{2/3} \quad \Rightarrow$$

Energy flux Eq.: $$\frac{d\varepsilon(k)}{dk} + \frac{\varepsilon(k)}{k}\,\frac{C_1C_2\,\phi\,\gamma(k)\tau_{\rm p}}{(1+\phi)[1+2\,\gamma(k)\tau_{\rm p}] + [\gamma(k)\tau_{\rm p}]^2} = 0 \Rightarrow$$

Simplicity $\Rightarrow$ effective analysis: $\varepsilon(k) = \dfrac{1}{\left[1+CJ(k)\right]^3}\,,\quad J(k) \simeq \dfrac{\phi}{4(1+\phi)}$

$$\times\left\{\ln\left[\frac{\delta(k\,L)^{2/3}+1+\phi-\sqrt{\phi(1+\phi)}}{\delta+1+\phi-\sqrt{\phi(1+\phi)}}\right] - \ln\left[\frac{\delta(k\,L)^{2/3}+1+\phi+\sqrt{\phi(1+\phi)}}{\delta+1+\phi+\sqrt{\phi(1+\phi)}}\right]\right\}\ .$$

Suppression & enhancement of turbulent energy by heavy particles

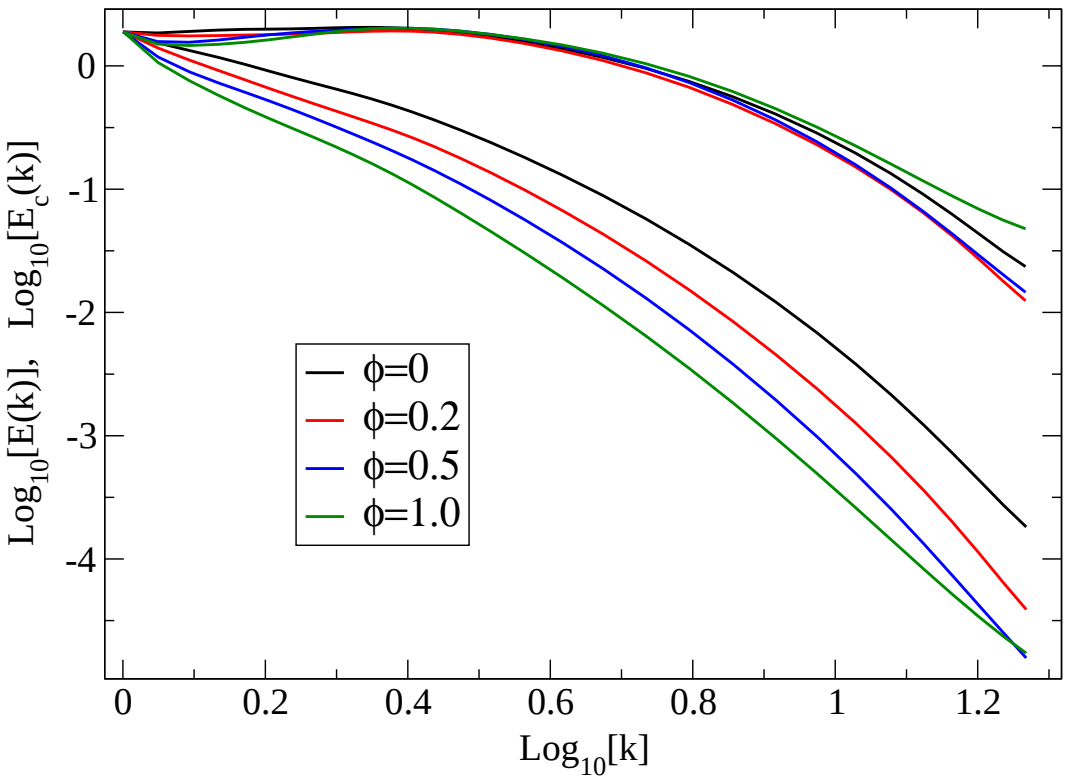


We found analytically: $R(kL) \equiv \dfrac{\text{Energy of } k\text{-eddies in suspension}}{\text{Energy of } k\text{-eddies in neat fluid}}$

$R(kL) < 1$ – suppression of **large** scales due to the fluid-particle friction,

$R(kL) > 1$ – enhancement of **small** scales due to the energy transfer from particles to the carrier fluid.

Suppression & enhancement of turbulent energy: theory *vs* DNS

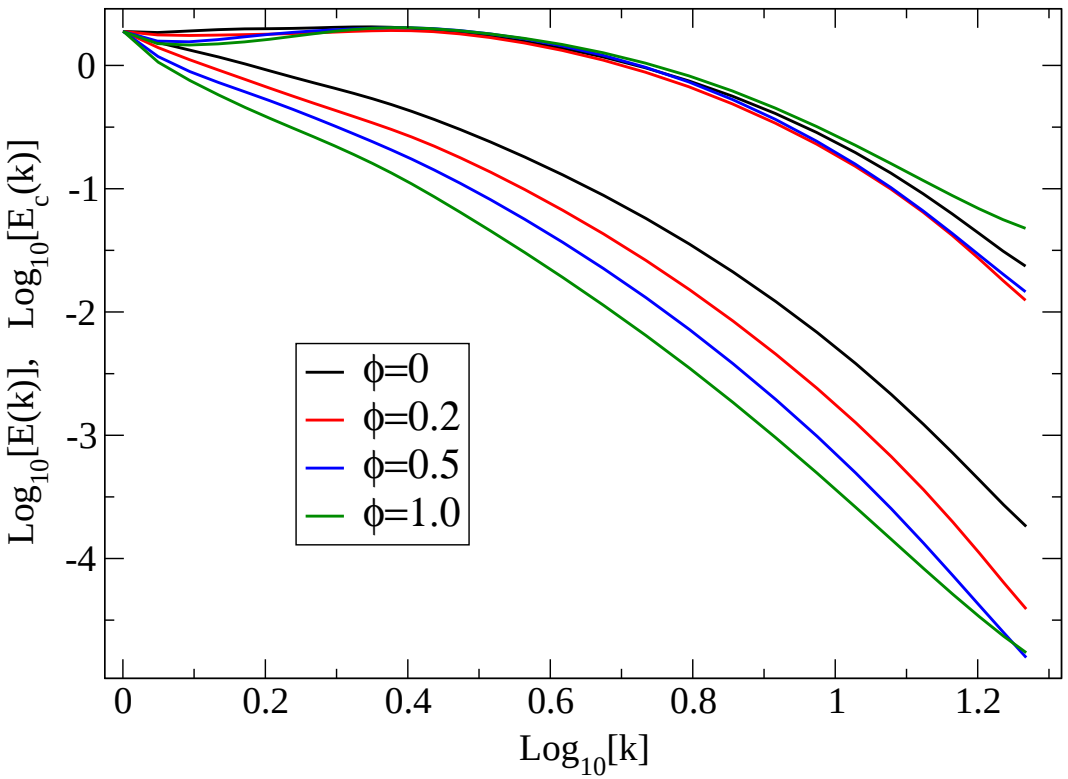


of Boivin-Simonin-Squires-'98 for $\phi = 0$ (black), 0.2, 0.5 & 1 – (color) lower lines.

Black upper line: "Compensated" spectrum for $\phi = 0$ (pure fluid): $\mathcal{E}_0(k) = E(k)(kL)^{5/3}$. In the inertial interval $\mathcal{E}_0(k)$ =const.

Color upper lines: Spectra $\mathcal{E}_\phi(k)$ for $\phi \neq 0$, "compensated" by our analytical solutions.

Suppression & enhancement of turbulent energy: theory *vs* DNS



of Boivin-Simonin-Squires-'98 for $\phi = 0$ (black), 0.2, 0.5 & 1 – (color) lower lines.

Black upper line: "Compensated" spectrum for $\phi = 0$ (pure fluid): $\mathcal{E}_0(k) = E(k)(kL)^{5/3}$. In the inertial interval $\mathcal{E}_0(k)$ =const.

Color upper lines: Spectra $\mathcal{E}_\phi(k)$ for $\phi \neq 0$, "compensated" by our analytical solutions.

Summary: Simple "One-fluid" model with K41 closure provide an **internally consistent analytical description** of the turbulence modification by particles: the dependencies of suppression or enhancement of turbulence on $\tau_{\rm p}\gamma_L$, ϕ, scale of eddies, and many other parameters.

2. Clustering of particles & droplets in *Space-homogeneous* turbulence of **Compressible**, **Two-phase** fluid, (water droplets in clouds, fuel droplets in internal combustion engines) [Elperin-Kleeorin-L-Liberman-Rogachevskii-Sokoloff-'02-04]

- Basic dynamical equation of motion: $\dfrac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) = D\Delta n \;\Rightarrow$

$$\Theta(t,\boldsymbol{r}) = n(t,\boldsymbol{r}) - \bar{n}\,, \quad \Rightarrow \quad \frac{\partial \Theta}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\Theta = -\Theta\,\mathrm{div}\,\boldsymbol{v} + D\Delta\Theta\ .$$

$n(t,\boldsymbol{r})$ – particle number density, D – coefficient of molecular (Brownian) diffusion

$\boldsymbol{v}(t,\boldsymbol{r})$ – particle velocity, $\neq \boldsymbol{u}(t,\boldsymbol{r})$ – fluid velocity,

$\mathrm{div}\,\boldsymbol{u} = 0$, however due to the particle inertia, $\mathrm{div}\,\boldsymbol{v} \neq 0$;

2. Clustering of particles & droplets in *Space-homogeneous* turbulence of **Compressible**, **Two-phase** fluid

- Basic dynamical equation of motion: $\dfrac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = D\Delta n \;\Rightarrow$

$$\Theta(t,\boldsymbol{r}) = n(t,\boldsymbol{r}) - \bar{n}\,, \quad \Rightarrow \quad \frac{\partial\Theta}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\Theta = -\Theta\,\mathrm{div}\,\boldsymbol{v} + D\Delta\Theta\ .$$

$n(t,\boldsymbol{r})$ – particle number density, D – coefficient of molecular (Brownian) diffusion

$\boldsymbol{v}(t,\boldsymbol{r})$ – particle velocity, $\neq \boldsymbol{u}(t,\boldsymbol{r})$ – fluid velocity,

$\mathrm{div}\,\boldsymbol{u} = 0$, however due to the particle inertia, $\mathrm{div}\,\boldsymbol{v} \neq 0$;

In the comoving with the center of a cluster reference frame:

$$\frac{\partial\Theta}{\partial t} + (\boldsymbol{w}\cdot\boldsymbol{\nabla})\Theta = -\Theta\,\mathrm{div}\,\boldsymbol{w} + D\,\Delta\,\Theta\,,$$

where $\boldsymbol{w}(\boldsymbol{\rho}\,|\,t,\boldsymbol{r}) =$ a particle velocity minus velocity of the cluster center $\boldsymbol{\rho}\,(t)$.

2. Clustering of particles & droplets in *Space-homogeneous* turbulence of **Compressible**, **Two-phase** fluid

- Basic dynamical equation of motion: $\dfrac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = D\Delta n \;\Rightarrow$

$$\Theta(t,\boldsymbol{r}) = n(t,\boldsymbol{r}) - \bar{n}\,, \quad \Rightarrow \quad \frac{\partial\Theta}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\Theta = -\Theta\,\mathrm{div}\,\boldsymbol{v} + D\Delta\Theta\ .$$

$n(t,\boldsymbol{r})$ – particle number density, D – coefficient of molecular (Brownian) diffusion

$\boldsymbol{v}(t,\boldsymbol{r})$ – particle velocity, $\neq \boldsymbol{u}(t,\boldsymbol{r})$ – fluid velocity,

$\mathrm{div}\,\boldsymbol{u} = 0$, however due to the particle inertia, $\mathrm{div}\,\boldsymbol{v} \neq 0$;

In the comoving with the center of a cluster reference frame:

$$\frac{\partial\Theta}{\partial t} + (\boldsymbol{w}\cdot\boldsymbol{\nabla})\Theta = -\Theta\,\mathrm{div}\,\boldsymbol{w} + D\,\Delta\,\Theta\,,$$

where $\boldsymbol{w}(\boldsymbol{\rho}\,|\,t,\boldsymbol{r}) =$ a particle velocity minus velocity of the cluster center $\boldsymbol{\rho}\,(t)$.

Suggested **Rigid-Cluster Model** $\Theta(\boldsymbol{\rho}|t,\boldsymbol{r}) \Rightarrow A_{\mathrm{Cl}}(t)\,\theta(\dfrac{|\boldsymbol{r}-\boldsymbol{\rho}|}{\ell_{\mathrm{Cl}}})$ reduces the problem to ODE for the cluster amplitude $A_{\mathrm{Cl}}(t)$, and

allows efficient analysis with physically transparent analytical results

One can consider separately:

- **Effect of particles inertia:** $\ominus\,\mathrm{div}\,w \quad \Rightarrow$ **"clustering" instability**

In the rigid-cluster model: $\mathrm{div}\,\boldsymbol{w}\,(\boldsymbol{\rho}\,|\,t,\boldsymbol{r}) \to b(t) \quad \Rightarrow \dfrac{\partial A_{\mathrm{Cl}}(t)}{\partial t} = -A_{\mathrm{Cl}}(t)\,b(t)$

$$\Rightarrow A_{\mathrm{Cl}}(t) = A_0\,\exp[-I(t)]\,,\;\; I(t) \equiv \int_0^t b(\tau)d\tau \;\; \Rightarrow \sum_{n=1}^{t/\tau_v} I_n\,,\;\; I_n(t) \equiv \int_{(n-1)\tau_v}^{n\tau_v} b(\tau)\,d\tau$$

$A_{\mathrm{Cl}}(t)$-Cluster amplitude, τ_v – correlation time of ℓ–eddies $\Rightarrow I_n(t) \sim$ random independent variables. Apply **Central Limiting Theorem:** $I(t) \sim S\sqrt{N}\zeta$, with $N = t/\tau_v$, $\quad S^2 = \langle I_n(t)^2\rangle_w \approx \langle b(t)^2\rangle_w\,\tau_v^2\,,\quad \zeta-$ random Gaussian variable, $\langle\zeta\rangle_{\mathrm{G}} = 0,\;\langle\zeta^2\rangle_{\mathrm{G}} = 1.$ Using $\langle\exp(A\,\zeta)\rangle_{\mathrm{G}} = \exp(A^2/2).$ After all these:

$$\mathcal{M}_q(t) \equiv \left\langle A_{\mathrm{Cl}}^q\right\rangle \propto \exp[+\gamma_{\mathrm{in}}(q)\,t]\,,\quad \gamma_{\mathrm{in}}(q) \sim \frac{1}{2}\langle\tau_v[\mathrm{div}\,\boldsymbol{w}(\boldsymbol{\rho}|t,\boldsymbol{r})]^2\rangle_w\,q^2 > 0\;.$$

One can consider separately:

- **Effect of particles inertia:** $\ominus\,\mathrm{div}\,w$ $\Rightarrow$ **"clustering" instability**

In the rigid-cluster model: $\mathrm{div}\,\boldsymbol{w}\,(\boldsymbol{\rho}\,|\,t,\boldsymbol{r}) \to b(t)$ $\quad\Rightarrow\quad \dfrac{\partial A_{\mathrm{cl}}(t)}{\partial t} = -A_{\mathrm{cl}}(t)\,b(t)$

$$\Rightarrow A_{\mathrm{cl}}(t) = A_0\,\exp[-I(t)]\,,\quad I(t) \equiv \int\limits_0^t b(\tau)d\tau$$

After all these:

$$\mathcal{M}_q(t) \equiv \left\langle A_{\mathrm{cl}}^q \right\rangle \propto \exp[+\gamma_{\mathrm{in}}(q)\,t]\,,\qquad \gamma_{\mathrm{in}}(q) \sim \frac{1}{2}\langle \tau_v[\mathrm{div}\,\boldsymbol{w}(\boldsymbol{\rho}|t,\boldsymbol{r})]^2\rangle_w\,q^2 > 0\ .$$

3 One can consider separately:

- **Effect of particles inertia:** $\ominus\,\mathrm{div}\,w$ $\Rightarrow$ **"clustering" instability**

In the rigid-cluster model: $\mathrm{div}\,\boldsymbol{w}\,(\boldsymbol{\rho}\,|\,t,\boldsymbol{r}) \to b(t)$ $\Rightarrow \dfrac{\partial A_{\mathrm{cl}}(t)}{\partial t} = -A_{\mathrm{cl}}(t)\,b(t)$

$$\Rightarrow A_{\mathrm{cl}}(t) = A_0\,\exp[-I(t)]\,,\quad I(t) \equiv \int_0^t b(\tau)d\tau$$

After all these:

$$\mathcal{M}_q(t) \equiv \left\langle A_{\mathrm{cl}}^q \right\rangle \propto \exp[+\gamma_{\mathrm{in}}(q)\,t]\,,\qquad \gamma_{\mathrm{in}}(q) \sim \frac{1}{2}\langle \tau_v[\mathrm{div}\,\boldsymbol{w}(\boldsymbol{\rho}|t,\boldsymbol{r})]^2\rangle_w\,q^2 > 0\ .$$

- **Effect of turbulent diffusion:** renormalization of molecular diffusion:

D by the effective turbulent diffusion D_{T} . $D \to D + D_T\,,\ D_T \sim \ell_{\mathrm{cl}} v_{\mathrm{cl}}/3 \Rightarrow$

$\Rightarrow$ After some simple consideration $\Rightarrow$

$$\gamma_{\mathrm{dif}}(q) \simeq -q\,\frac{D_T}{\ell_{\mathrm{cl}}^2} \simeq -\frac{q}{3}\,\frac{v_{\mathrm{cl}}}{\ell_{\mathrm{cl}}} \simeq -\frac{q}{3\,\tau_{\mathrm{cl}}}\,,$$

τ_{cl} turnover time of the ℓ_{cl}-eddies

- **Estimate of the growth rate of clustering instability**

$$\mathcal{M}_q(t) = \langle |A_{\rm cl}(t)| \rangle = \mathcal{M}_q(0)\, \exp(\gamma_q\, t)\,, \qquad \gamma_q \simeq \gamma_{\rm dif}(q) + \gamma_{\rm in}(q)\,,$$

$$\gamma_q \simeq -\frac{q}{3\,\tau_{\rm cl}} + \frac{q^2}{2}\langle \tau_v [{\rm div}\, \boldsymbol{w}]^2 \rangle \simeq -\frac{q}{3\,\tau_{\rm cl}} + \frac{q^2}{2\,\tau_{\rm cl}}\,\sigma$$

$\sigma \equiv \dfrac{\langle [{\rm div}\, \boldsymbol{w}]^2 \rangle}{\langle |\boldsymbol{\nabla} \times \boldsymbol{w}|^2 \rangle} \simeq \left(\dfrac{\rho_{\rm p}}{\rho_{\rm f}}\right)^2 \left(\dfrac{a}{\eta}\right)^4 \equiv \left(\dfrac{a}{a_*}\right)^4$ compressibility parameter,
η – Kolmogorov microscale, a- particle radius.

For water droplets in the atmosphere $\eta \approx 1{\rm mm}, \quad a_* \approx 30\mu{\rm m}$;

For fuel droplets in a car diesel engines $\eta \approx 10\mu{\rm m}, \quad a_* \approx 40\mu{\rm m}$.

- **Estimate of the growth rate of clustering instability**

$$\mathcal{M}_q(t) = \langle |A_{\rm cl}(t)| \rangle = \mathcal{M}_q(0)\, \exp(\gamma_q\, t)\,, \qquad \gamma_q \simeq \gamma_{\rm dif}(q) + \gamma_{\rm in}(q)\,,$$

$$\gamma_q \simeq -\frac{q}{3\,\tau_{\rm cl}} + \frac{q^2}{2}\langle \tau_v [{\rm div}\, \boldsymbol{w}]^2 \rangle \simeq -\frac{q}{3\,\tau_{\rm cl}} + \frac{q^2}{2\,\tau_{\rm cl}}\,\sigma$$

$\sigma \simeq \left(\frac{a}{a_*}\right)^4$ compressibility parameter, a- particle radius.

For water droplets in the atmosphere $\eta \approx 1{\rm mm}, \quad a_* \approx 30\mu{\rm m}$;

For fuel droplets in a car diesel engines $\eta \approx 10\mu{\rm m}, \quad a_* \approx 40\mu{\rm m}$.

Summary: For $a > a_*$ homogeneous particles/droplets distribution is unstable $\Rightarrow$ clusterization. In particular, the probability of particle-particle collision $\propto \left\langle n^2 \right\rangle \propto \mathcal{M}_2(t) \propto \exp(\gamma_2\, t)$ and finally increases in orders of magnitude. Thus, the clustering instability of $\mathcal{M}_2$ has reach consequences for dynamics of water droplet in turbulent clouds, fuel droplets in diesel engines, etc.

3. Simple model of Turbulent Boundary Layer: Space-inhomogeneous, Near-wall turbulence of *Incompressible*, *One-phase* fluid (Atmospheric TBL, channel & pipe high-$\mathcal{R}e$ flows) [L'vov-Pomyalov-Tiberkevich-'03]

In our simple model of TBL:

– Space energy transfer is ignored with respect of local energy dissipation ⇒Only **Local** (in $\boldsymbol{r}$) **Mechanical Balance** and **Local Energy Balance** are accounted for.

– Space derivatives are estimated via distance to the wall.

Navier-Stokes Eq. ⇒ Exact **Local Mechanical Balance** Eq.:

$$\nu \frac{dV(y)}{dy} + W(y) = p' L \,,$$

$$W(y) \equiv -\langle u_x u_y \rangle \;,\; p' \equiv -\frac{dp}{dx} = \text{const} \,.$$

Kolmogorov-41 inspired Model for **Balance of Local Energy $K(y)$:**

$$\left[\nu\left(\frac{a}{y}\right)^2 + \frac{b\sqrt{K(y)}}{y}\right] K(y) = W(y)\frac{dV(y)}{dy} \,,$$

$$K(y) \equiv \frac{1}{2}\left\langle |\boldsymbol{u}(\boldsymbol{r},t)|^2 \right\rangle \,, \quad a,\; b \sim 1$$

• **Analytical solution:** We have two Eqs. for the three objects:

Mean shear, $S(y)y \equiv dV(y)/dy$, Kinetic energy, $K(y) = \frac{1}{2}\left\langle u^2 \right\rangle$ and Reynolds stress, $W(y) \equiv -\langle u_x u_y \rangle$:

- For mechanical momentum: $[\nu_0 + \nu_\mathrm{p}(y)]\, S(y) + W(y) = p'\, L\,,$ (1)
- Kin. energy: $\{[\nu_0 + \nu_\mathrm{p}(y)](a/y)^2 + b\sqrt{K(y)}/y]\}K(y) = W(y)S(y)\,,$ (2)
- Introduce simple TBL closure: $W(y) = c_\mathrm{N}^2 K(y)\ .$ (3).

Solve quadratic eq. (1)-(3) for $S(y)$ & integrate. **The result**:

For $y^+ \le y_v^+ = a/c_\mathrm{N}$: $V^+ = y^+\ .$ (4a)

For $y^+ \ge y_v^+$:

$$V^+(y^+) = \frac{1}{\kappa_\mathrm{K}} \ln Y(y^+) + B - \Delta(y^+)\,, \quad B = 2y_v^+ - \frac{1}{\kappa_\mathrm{K}} \ln\left[\frac{e\,(1 + 2\,\kappa_\mathrm{K} y_v^+)}{4\kappa_\mathrm{K}}\right],$$

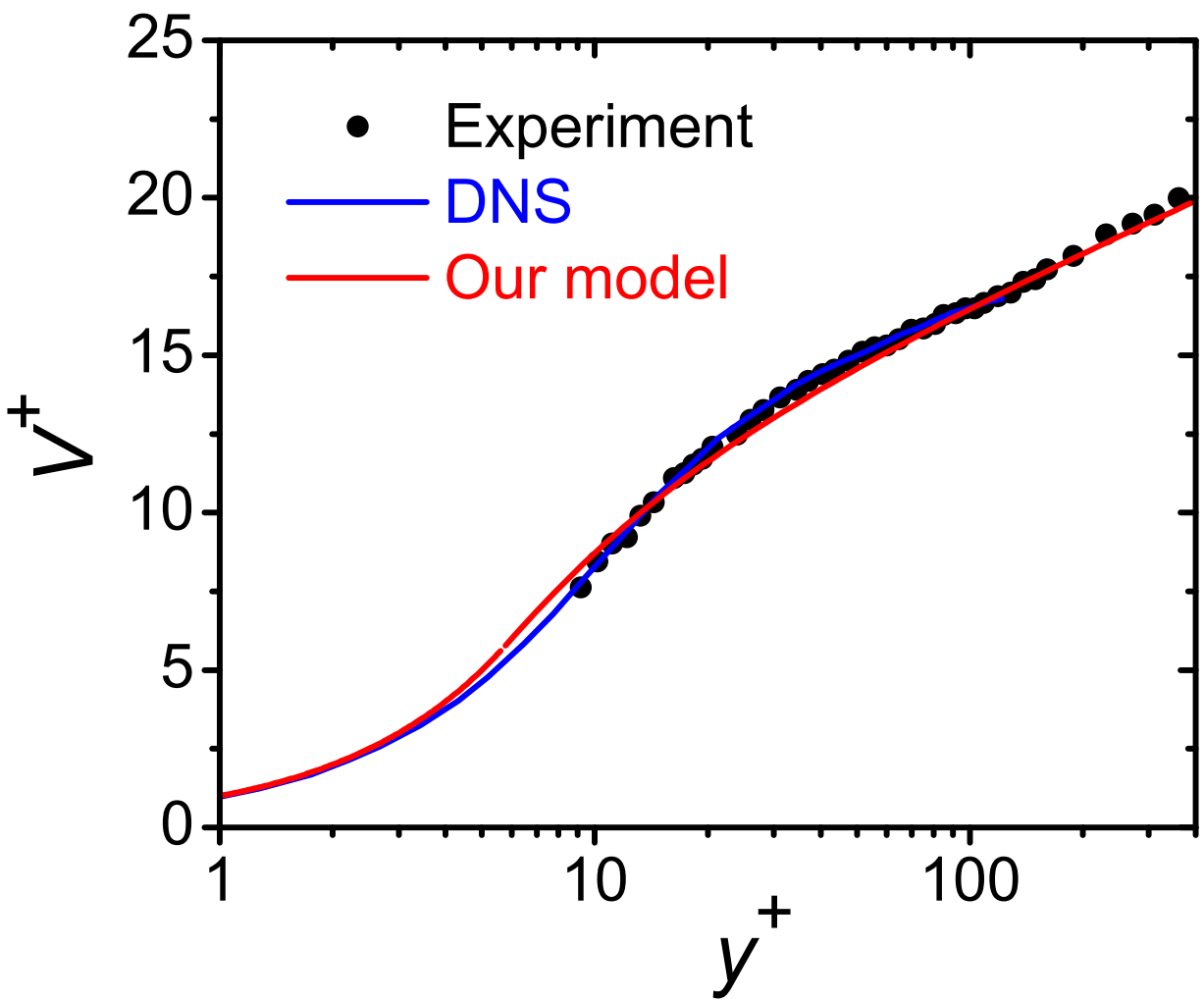
$$Y(y^+) = [y^+ + \sqrt{y^{+2} - y_v^{+2} + (2\kappa_\mathrm{V})^{-2}}]/2\,, \qquad \kappa_\mathrm{K} = c_\mathrm{N}/b\,, \qquad (4b)$$

$$\Delta(y^+) = \frac{2\,\kappa_\mathrm{K}^2 y_v^{+2} + 4\kappa_\mathrm{K}[Y(y^+) - y^+] + 1}{2\kappa_\mathrm{K}^2 y^+}\ .$$

Two fit parameters: κ_N & B .

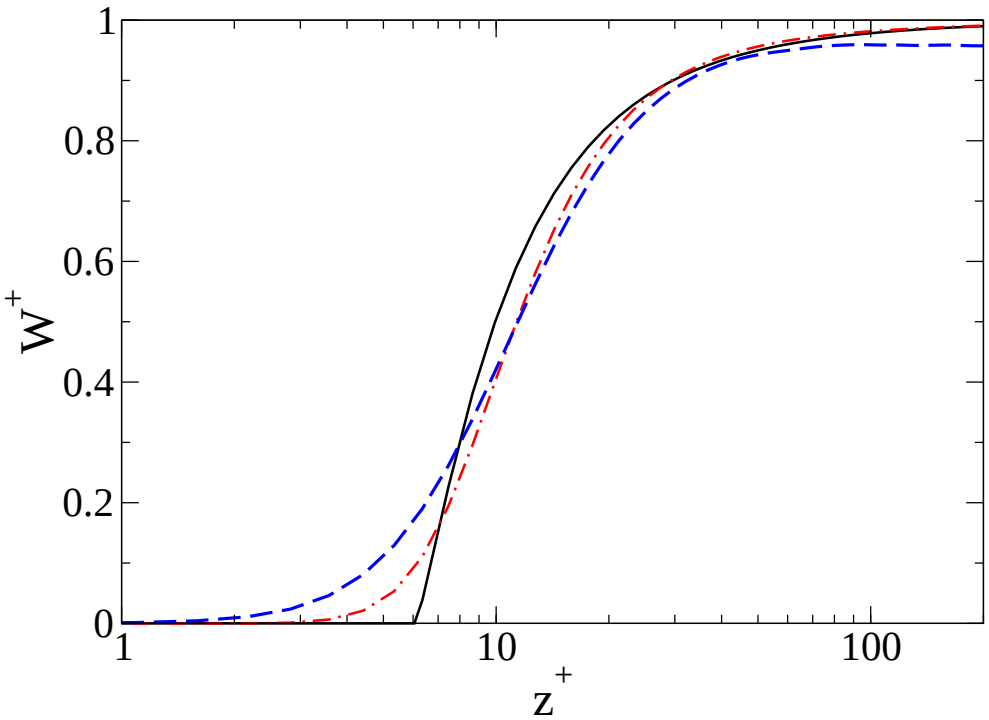
• Comparison of analytical profile (4) with experiment & numerics

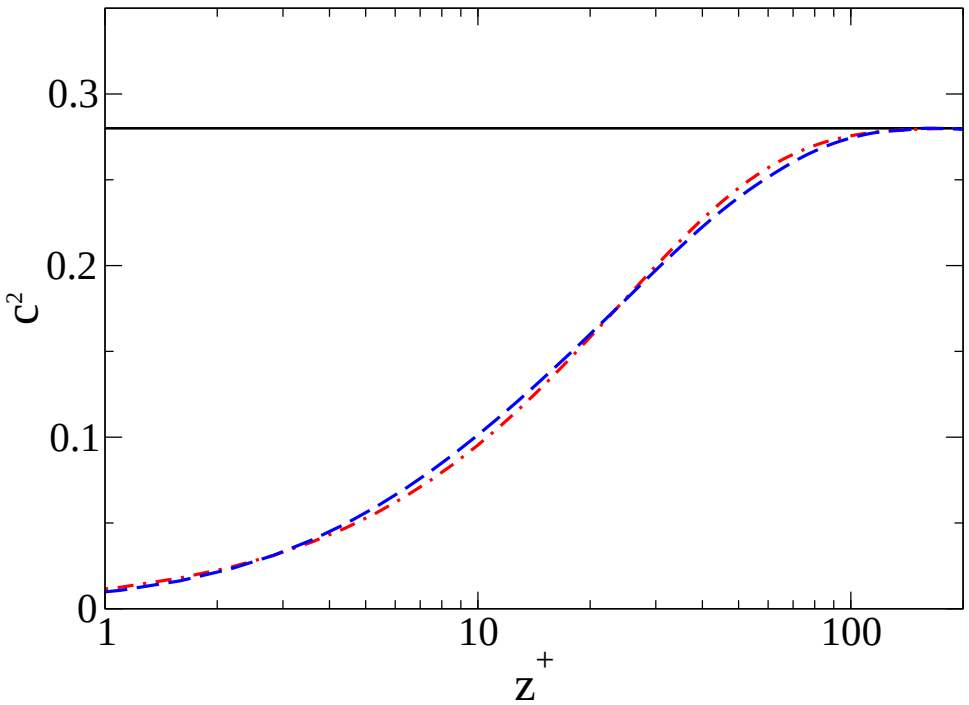
– Prandtl-Karman profile $V^+ = 2.5 \ln y^+ + 5.5$ consts. are used in (4)



One sees that our simple Algebraic Reynolds-stress model, based on the *exact* balance of mechanical momentum, K41 inspired *model equation* for the local energy balance, and simple TBL closure for W/K gives analytical, semi-quantitative description of the mean velocity profile.

.

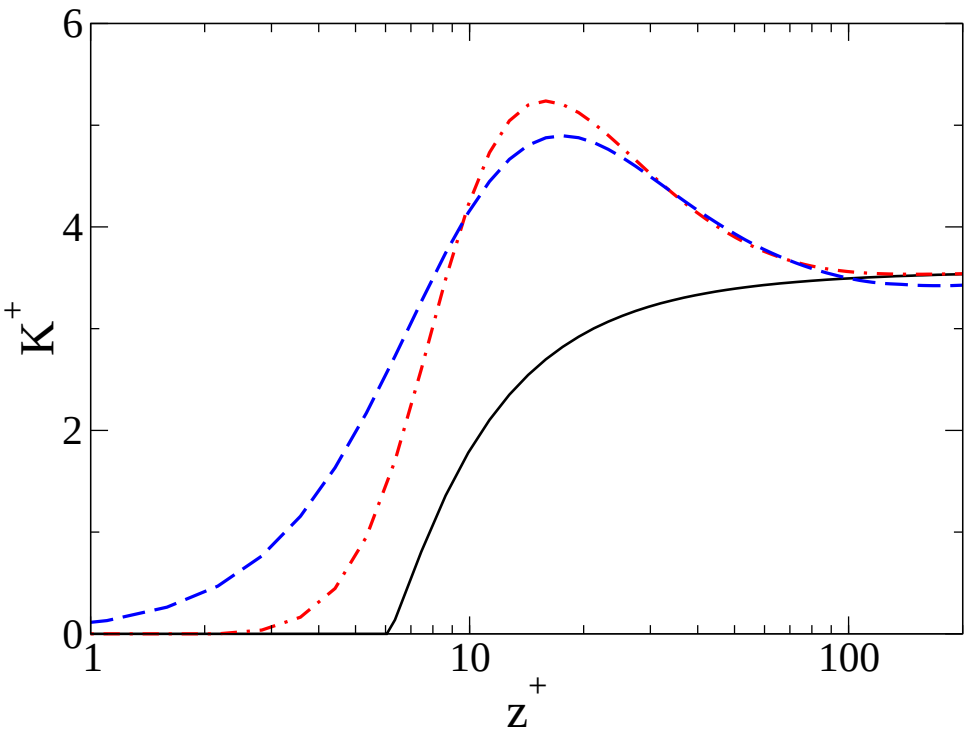


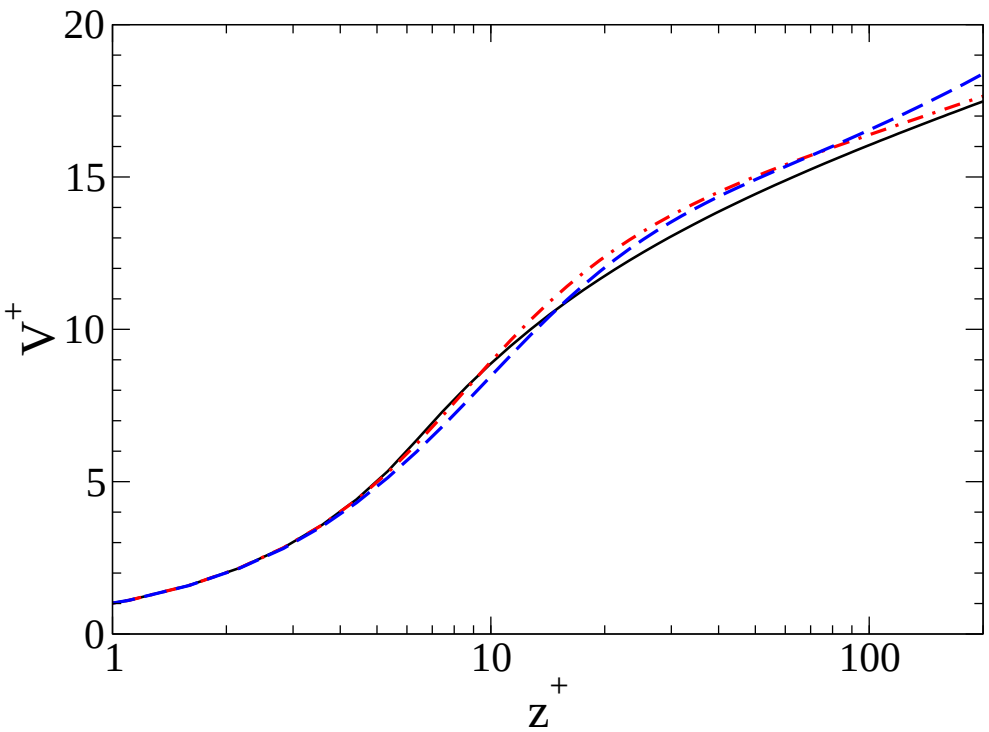


Left: Profiles of the Reynolds stress $W^+(z^+)$: Black solid line – simple analytical model with $c_{\mathrm{N}}^2 = 0.28$; Blue dashed line – DNS data. Red dash-dotted line – Improved model with fit function (*) for $c(z^+)$.

Right: Function $c^2(z^+) = W(z^+)/K(z^+)$**:** Blue dashed line – DNS data, Red dash-dotted line – suggested fit for $c^2(z^+)$:

$$c^2(z^+) = 0.28\left[1 - \exp\left(-\frac{z^+}{24}\right)\right] \quad . \qquad (*)$$





Black solid lines – simple analytical model with $c_{\mathrm{N}}^2 = 0.28$,

Red dash-dotted lines – Improved model with fit function (*) for $c(z^+)$

Blue dashed lines – DNS data of R.G. Moser, J. Kim & N.N. Mansour.

Conclusion:

Improvement $c(z^+)$ $\neq$const. does not effect the mean velocity profile, slightly improves Reynolds stress and important only for the kinetic energy.

Summary:

- We suggested simple Algebraic Reynolds-stress model, based on
  - i) *Exact* balance of mechanical momentum,
  - ii) K41 inspired *model equation* for the local energy balance, and
  - iii) Simple TBL closure for W/K.
- The model gives analytical, semi-quantitative description of the mean velocity profile, and profiles of the Reynolds stress and kinetic energy
- The physical transparency and simplicity of the model allows its generalization for turbulently flowing suspension, laden with
  - i) polymers, leading to a theory of drag reduction by dilute additives of flexible polymers, (VL, I.Procaccia, A.Pomyalov, V.Tiberkevich, PRL, 2004)
  - ii) microbubbles (VL, AP, IP, VT PRL, submitted)
  - iii) heavy particles (atmospheric TBL over stormy see and deserts), etc.