TURBULENCE IN MAGNETIZED PLASMAS AND THE ISSUE OF ZONAL MODES

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OUTLINE

- **Plasmas**
  - Definition of a plasma
  - Mathematical description of plasmas
  - Plasma turbulence (vs fluid turbulence)
- **Turbulent generation of zonal modes**
  - Zonal mode generation in magnetized plasma turbulence
  - Zonal mode generation in quasi-geostrophic turbulence

A broad class of systems exhibits spectral condensation phenomena. One example is magnetized plasma turbulence, where *large-scale zonal modes* ($k_y = 0$) are generated by an inverse energy transfer from unstable drift-waves, *despite the fact that the relevant nonlinearity supports a direct energy cascade.*

PLASMAS

- A plasma is a gas with part or all the atoms dissociated into positive ions and negative electrons: $\Rightarrow$ a collection of ***discrete charged particles*** moving in a *self-consistent* e.m. field: the fields affect the particle orbits, and the particle orbits affect the field $\Rightarrow$ *nonlinear system*.

- Differently from a nonionized gas, charged particles generate e.m. fields and thus can act simultaneously producing ***collective phenomenon***.

Depending on the plasma parameters, the relevant point of view is either that of a collection of discrete particles, or that of a fluid supporting currents, or that of both.

In this talk I will consider plasmas that are:

- *fully-ionized* (all atoms are ionized)
- *neutral* (equal number of singly-charged ions and electrons, with total charge equal to zero)
- *high-temperature* (the potential energy of a typical particle due to its nearest neighbor is much less that its kinetic energy - *weakly-coupled plasmas*)
- *strongly-magnetized* (the dynamics is quasi-2D and closely related to neutral fluids)
- *classical* (no relativity and/or quantum mechanics)

These kind of plasmas occur naturally in many physical settings [*stellar interiors and atmospheres*, in *gaseous nebulae*, in *interstellar hydrogen*], and are routinely created in *laboratories studying fusion plasma physics*.

DESCRIPTION OF PLASMAS

KLIMONTOVICH EQUATION (& Maxwell eqns)

↓ *ensemble average*

PLASMA KINETIC EQUATION —*neglect collisions*→ **VLASOV EQUATION**

↓ *take V−moment*

TWO−FLUID THEORY —*various approximations*→ **REDUCED TWO−FLUID THEORIES**

↓ *combine*

ONE−FLUID (MHD) THEORY —*various approximations*→ **REDUCED MHD EOUATIONS**

Klimontovich equation

It is the fundamental plasma kinetic equation: it describes the plasma by taking into account *the motion of all particles*:

$$\frac{\partial N_s(\mathbf{x},\mathbf{v};t)}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}} N_s + \frac{q_s}{m_s}\left(\mathbf{E}^m(\mathbf{x};t) + \frac{\mathbf{v}}{c}\times\mathbf{B}^m\right)\cdot\boldsymbol{\nabla}_{\mathbf{v}} N_s = 0$$

where

$$N_s(\mathbf{x},\mathbf{v};t) = \sum_{i=1}^{N_0} \delta[\mathbf{x}-\mathbf{X}_i(t)]\delta[\mathbf{v}-\mathbf{V}_i(t)] \qquad \textbf{spiky function!}$$

$(\mathbf{x},\mathbf{v})$: 6D Eulerian phase space coordinates
$(\mathbf{X}_i,\mathbf{V}_i)$: 6D Lagrangian particle coordinates

Plasma kinetic equation

We are usually not interested in the exact motion of all the particles in the plasma, but rather in *certain average or approximate characteristics*:

$$N_s(\mathbf{x}, \mathbf{v}; t) \xrightarrow{\text{average}} \langle N_s(\mathbf{x}, \mathbf{v}; t)\rangle \equiv f_s(\mathbf{x}, \mathbf{v}; t) \quad \textbf{smooth}!$$

where $\langle \cdots \rangle$ is an ensemble average over an infinite number of realizations of the plasma.

$f_s(\mathbf{x}, \mathbf{v}; t)$ *is the (very large) number of particles of species s per unit configuration space per unit velocity space.*

$$\text{Setting :} \qquad N_s(\mathbf{x},\mathbf{v};t) = \overbrace{f_s(\mathbf{x},\mathbf{v};t)}^{\text{smooth}} + \overbrace{\delta N_s(\mathbf{x},\mathbf{v};t)}^{\text{spiky}}$$

$$\mathbf{E}^m(\mathbf{x};t) = \mathbf{E}(\mathbf{x};t) + \delta\mathbf{E}(\mathbf{x};t)$$

$$\mathbf{B}^m(\mathbf{x};t) = \mathbf{B}(\mathbf{x};t) + \delta\mathbf{B}(\mathbf{x};t)$$

we obtain the *plasma kinetic equation*

$$\overbrace{\frac{\partial f_s(\mathbf{x},\mathbf{v};t)}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_s}^{\text{collective effects (smooth)}}$$

$$= \underbrace{-\frac{q_s}{m_s}\left\langle\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\mathbf{v}\,\delta N_s\right\rangle}_{\text{discrete-particle (collisional) effects (spiky)}}$$

When $|RHS|/|LHS| \simeq 1/\Lambda \ll 1$ we can neglect collisions: Vlasov eq. [collective effects + kinetic effects (particle-wave resonances)].

Two-fluid equations ($\omega \ll \nu_{\mathrm{ei}}$)

The distinct feature of a plasma as a continuous medium lies in the different responses of the electrons and ions, which induce collective electromagnetic fields. A set of *two-fluid equations derived by the kinetic equation in the collisional limit* is thus necessary

$$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s \mathbf{V}_s) = 0 \qquad (s = e, i)$$

$$n_s m_s \frac{D\mathbf{V}_s}{Dt} \pm e_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) + \nabla p_s + \nabla\cdot\mathbf{\Pi}_s^d = \pm\mathbf{R}_{ei}$$

$$\frac{3}{2} n_s \frac{DT_s}{Dt} + p_s \nabla\cdot\mathbf{V}_s = -\nabla\cdot\mathbf{q}_s - \mathbf{\Pi}_s^d : \nabla\mathbf{V}_s + Q_s$$

$$\varepsilon_0 \nabla\cdot\mathbf{E} = \sum_{s=i,e} e_s n_s \ , \qquad \frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \ , \qquad \nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0 \mathbf{J} = \mu_0 \sum_{s=i,e} e_s n_s \mathbf{V}_s$$

16 equations in 16 unknowns: $n_e, n_i, \mathbf{V}_e, \mathbf{V}_i, T_e, T_i, \mathbf{E}, \mathbf{B}$

Plasma dynamics can have *many characteristic time scales* (associated to various wave perturbations, time-variations of background quantities, collision frequency between species, etc.) and *length scales* (e- and i-gyroradius, collision mean free paths, etc.).

- The existence of different time and length scales can be used to construct *reduced set of equations* containing a fewer number of unknowns. (For example, a two-time scale approach can *eliminate high-frequency oscillations* associated with Alfvén waves $\Rightarrow$ get equations to analyze the slower time-scale evolution of pressure-gradient-driven instabilities, or electrostatic perturbations.)

THE PLASMA TURBULENCE PROBLEM

Plasmas are turbulent

- Almost all plasmas are *linearly unstable* to a variety of waves
- The growth of the oscillations is *nonlinearly saturated*, leaving a state of *random fluctuations, or turbulence*
- Turbulence leads to *enhanced transport* effects

Approach

1. Identification of *linear instability*
2. Identification of the *saturation mechanism*, and calculation/detection of the steady-state *fluctuation levels* (e.g. $\langle \tilde{E}^2 \rangle$, $\langle \tilde{n}^2 \rangle$, etc.)
3. Calculation of *transport coefficients* as functions of $\langle \tilde{E}^2 \rangle$, etc.

Step 2. is the real problem.

FLUID VS PLASMA TURBULENCE

The term "turbulence" is used for plasmas in a *broader sense* than in conventional hydrodynamics:

- *Hydrodynamic turbulence* consists of a large number of mutually interacting *eddies*;
  Plasma turbulence consists of both *eddies* and *oscillations*.
- *Hydrodynamic eddies* have small relative velocities and therefore interact over a *long time* (strong turbulence);
  Wave packets in a plasma can interact over *short times* and can separate from one another *over large distances* (turbulence can range from very weak to strong).

Similarities

- Although most fluid research is 3D, many *important fluid applications are 2D (geophysics)*, and this has strong similarities to *plasmas confined by a strong magnetic field* (which shapes the fluctuations to have a very long correlation length along the field lines and are quasi-2D).
- In both fields most work has been done on *quadratic nonlinearities* [NS: $(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}$; PL: $(\mathbf{V}_E\cdot\boldsymbol{\nabla})\mathbf{V}$ or $(\mathbf{V}_E\cdot\boldsymbol{\nabla})n$].
- The *closure problem is common*: moment-based statistical closure approximations provide a way of expanding a cumulant of some order (say, 3) in terms of lower-order cumulants

Differences (more numerous!)

- A plasma supports a much larger number of oscillations *("the plasma wave zoo")*, and most of them are made *unstable* by inhomogeneities in the plasma parameters (n, T), in the e.m. fields, and in velocity space.

- *Wave-particle resonance important in plasmas* $\Rightarrow$ *need to view the plasma as a collection of interacting waves and particles:*
  - The linearized NS propagator describes viscous dissipation:

$$G_{NS}^{\rm lin} = \left[-i(\omega + ik^2\mu)\right]^{-1}$$

  - The linearized Vlasov propagator describes both collective wave effects and ballistic particle streaming (broaden by turbulence):

$$G_L^{\rm lin} \xrightarrow{\epsilon\rightarrow 0} \left[i\mathcal{P}\left(\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}}\right) + \pi\delta(\omega - \mathbf{k}\cdot\mathbf{v})\right]\delta(\mathbf{v} - \mathbf{v}')$$

- In plasmas, *two facts conspire against the development of a well-developed (Kolmogorov) inertial range:* (1) the *abundance of linear dissipation mechanisms* (Landau damping), which limits the minimum excitable scale, and (2) the *nature of forcing*, often better modeled by a self-limiting linear growth-rate term than by an external, localized, random forcing.
- Spectral transfer in plasmas: *non self-similar, anisotropic, highly nonlocal.*

DRIFT-WAVE TURBULENCE IN MAGNETIZED PLASMAS

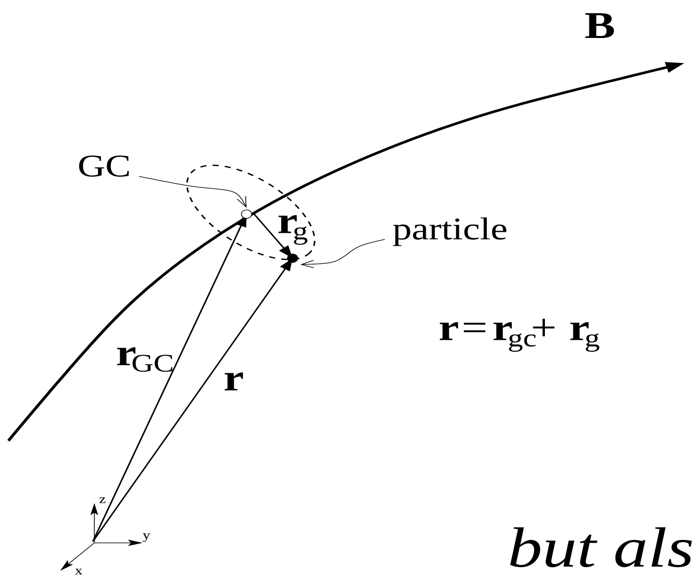
I will present results (*generation of* $k_y = 0$ *modes*) derived from a reduced ***two-field model (in*** $\mathbf{n}_e$ ***and*** ϕ***)*** of ***low-frequency electrostatic turbulence*** in a ***strongly magnetized plasmas***, relevant to tokamak physics.

- The basic linear oscillations are **drift-waves** (very similar to Rossby waves in the geophysical context).
- These waves are destabilized by the presence of **electrons trapped in the magnetic potential**.
- The turbulence is fed by the the **unstable drift-waves in the intermediate region of k-space**.

Charged particle motion

$$m\frac{d^2\mathbf{r}(t)}{dt^2} = q\mathbf{E} + q\frac{d\mathbf{r}(t)}{dt}\times\mathbf{B}$$ Lorentz force

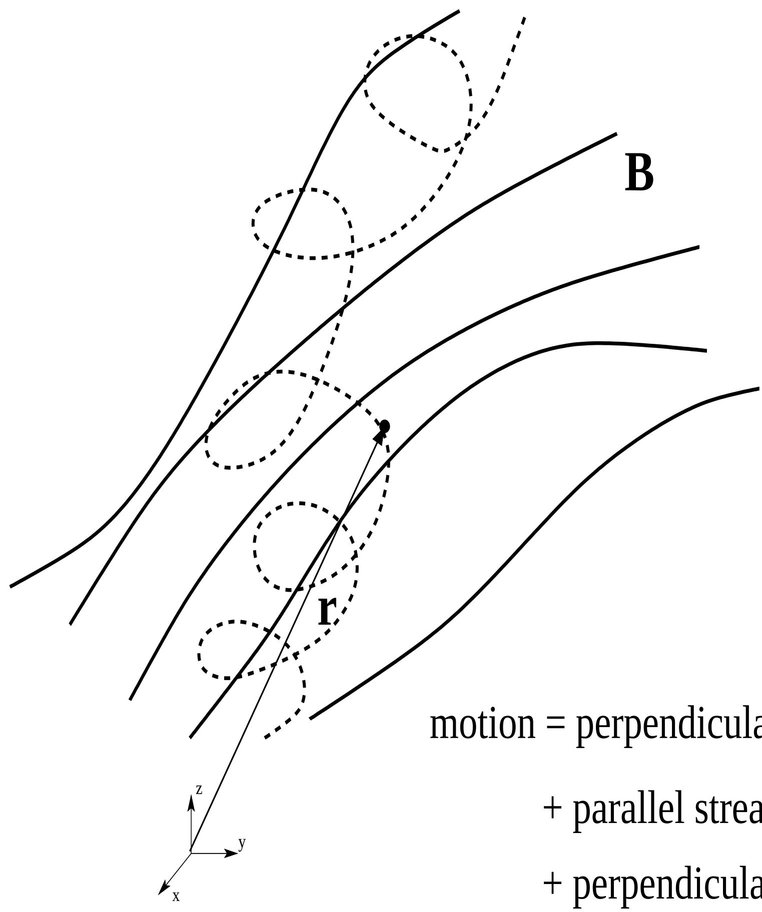
induces *gyrations*, *parallel streaming*,



but also

... *$E \times B$ and other drifts perpendicular to* $\mathbf{B}$:

$$0 = q\mathbf{E} + q\mathbf{v}\times\mathbf{B} \quad \xrightarrow{\times\mathbf{B}} \quad \mathbf{v}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

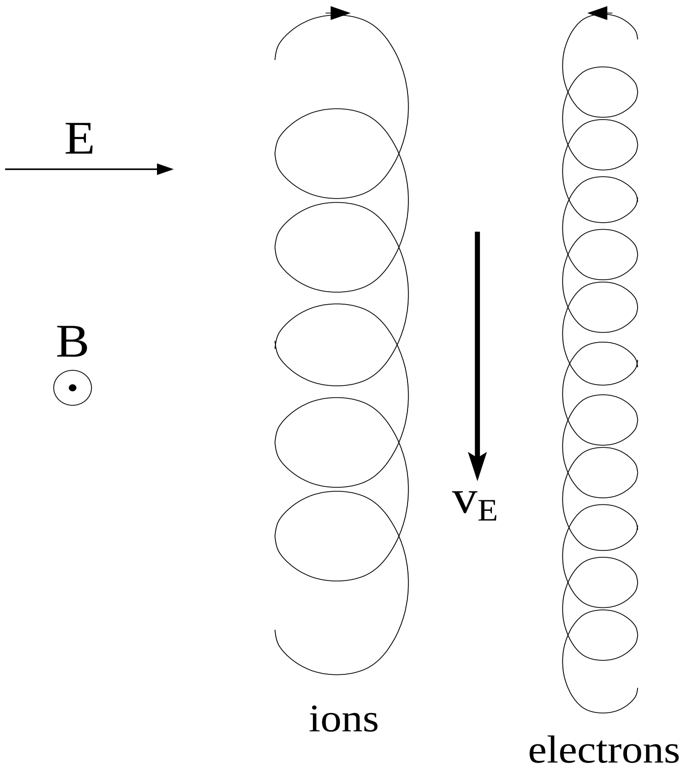


motion = perpendicular gyration

+ parallel streaming

+ perpendicular drift

Physics of $\mathbf{E} \times \mathbf{B}$ drift



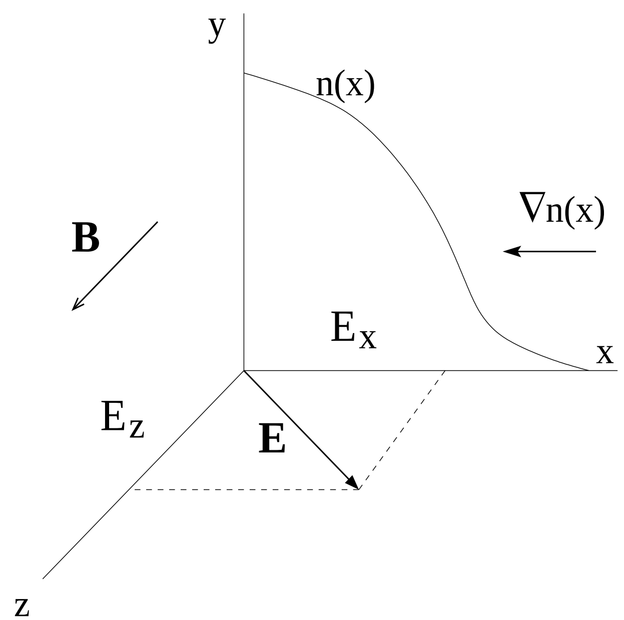
- First half of the orbit: ion gains energy from electric field, $F_i = +e\mathbf{E}$, hence increasing its $v_\perp$ and $r_L \propto v_\perp / B_0$
- Second half-cycle: it loses energy ($\mathbf{E}$ contrary to the direction of motion) and decreases its r_L.
- *Difference in r_L causes ion to drift*

$\mathbf{E} \times \mathbf{B}$ and diamagnetic drifts

Momentum equation for species s (i.e., fluid picture),

$$m_s n_{s,0} \left[\frac{\partial \mathbf{V}_s}{\partial t} + (\mathbf{V}_s \cdot \boldsymbol{\nabla}) \mathbf{V}_s \right] = q_s n_{s,0} (\mathbf{E}_0 + \mathbf{v}_s \times \mathbf{B}_0) - \boldsymbol{\nabla} p_{s,0} \ .$$

Here, $\mathbf{E}_0$ and $\mathbf{B}_0$ are uniform, and $n_{s,0}$ and $p_{s,0}$ have a gradient in the x-direction:



- Ratio of linear inertial term and magnetic Lorentz force

$$\frac{m_s n_{s,0}(\partial \mathbf{V}_s/\partial t)}{q_s n_{s,0}\mathbf{V}_s \mathbf{B}_0} \simeq \frac{\omega}{\omega_c} = \omega \frac{m_s}{eB_0} \; .$$

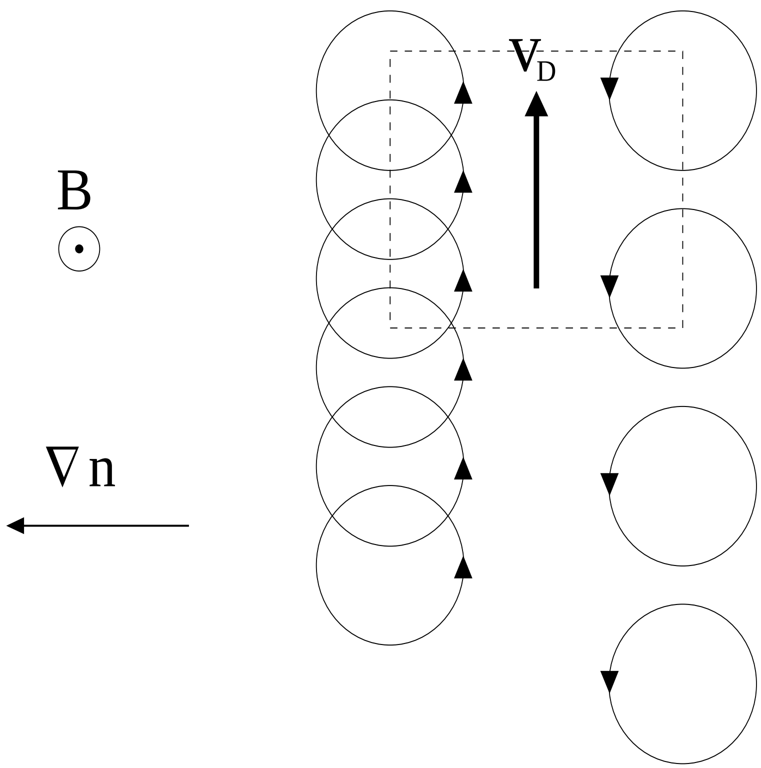
- *Neglect linear inertial term* (drifts slow compared to gyration)
- *Neglect nonlinear inertial term* (linear problem)
- Equation of motion reduces to

$$0 = q_s n_{s,0}(\mathbf{E}_0 + \mathbf{V}_s \times \mathbf{B}_0) - \boldsymbol{\nabla} p_{s,0} \; .$$

- Taking $\times \mathbf{B}_0$ we solve for $\mathbf{V}_{s\perp}$:

$$\mathbf{V}_{s\perp} = \overbrace{\frac{\mathbf{E}\times\mathbf{B}}{B_0^2}}^{E \,\times\, B \text{ drift}} - \overbrace{\frac{\boldsymbol{\nabla} p_{0s} \times \mathbf{B}_0}{q_s n_{s,0} B_0^2}}^{\text{diamagnetic drift: } \mathrm{v_D}} \; .$$

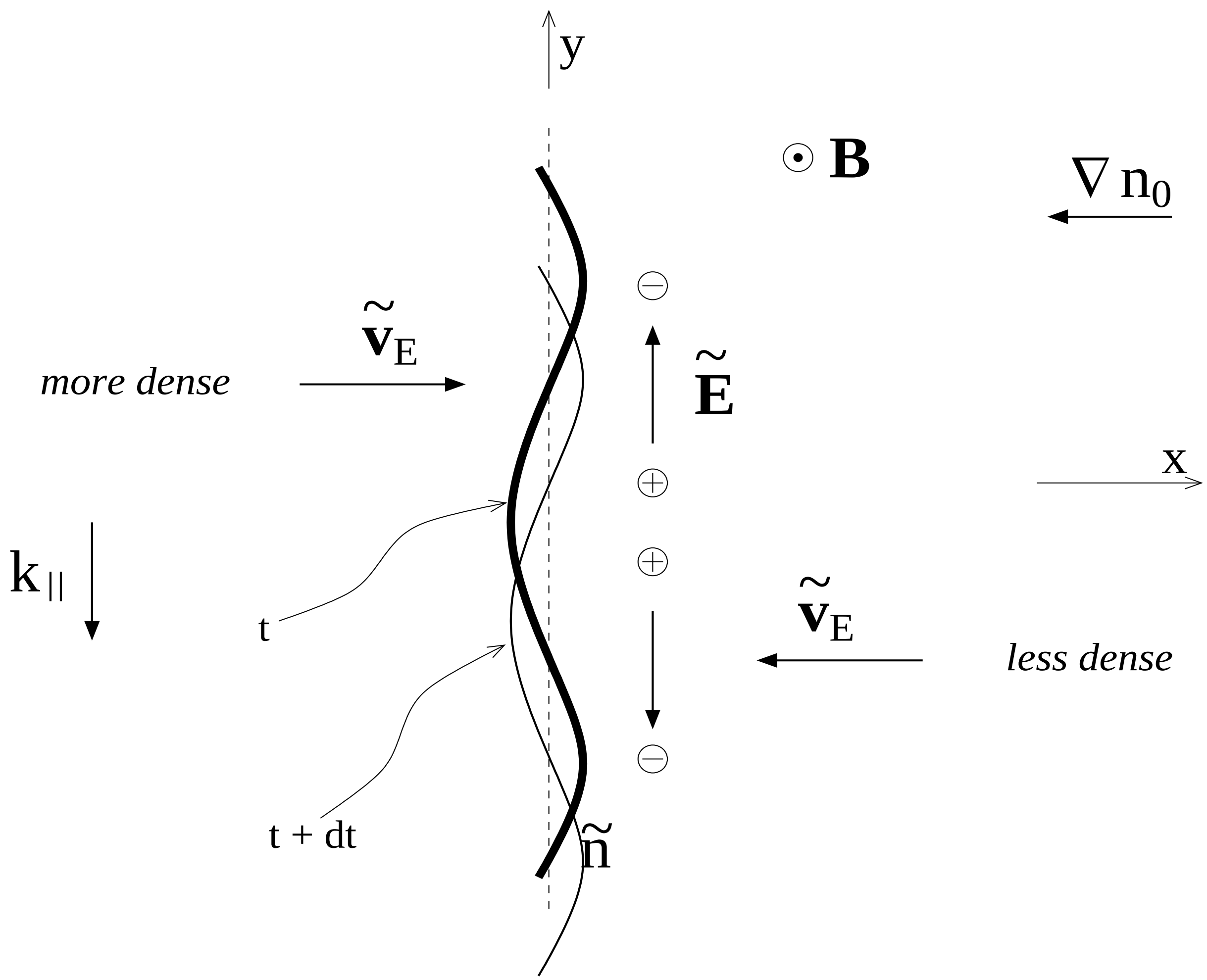
Physics of the diamagnetic drift ($\mathbf{v_{D,s} = -T_s \nabla n_{0,s} \times B_0 / q_s n_{0,s}}$



- Through any fixed volume element there are more ions moving upward then downward (upward-moving ions come from region of higher density).
- Therefore, there is a *fluid drift* perpendicular to $\boldsymbol{\nabla} n_0$ and $\mathbf{B}_0$ (even though the guiding centers are stationary!)

Drift waves

At time t we introduce a small electron density perturbation λ_y:



Dispersion relation for drift waves

$$\omega = v_{D,e} k_y$$

- When $\tilde{\mathbf{n}}_{\mathbf{e}}$ **and** $\tilde{\phi}$ **are exactly in phase**, the change in the density perturbation due to $\mathbf{v}_E$ is exactly 90^0 ahead of the original density perturbation: ***purely oscillatory wave***.
- In real magnetically confined plasmas, several effects (collisions, inertia, *trapping*, etc.) limit electron mobility introducing a ***lag between*** $\tilde{\mathbf{n}}_{\mathbf{e}}$ ***and*** $\tilde{\phi}$: density perturbation becomes self-reinforcing ($\mathbf{v}_E$ is outward when the plasma has already been shifted outward, and vice-versa): ***wave becomes unstable***.

Nonlinear equations for stable drift-waves

- *Cold ions*, $T_i \ll T_e$ $(\omega_{c,i} = eB_0/m_i)$, *with inertia*:

$$\mathbf{v}_{i\|} = 0 \; , \qquad \mathbf{v}_{i\perp} = \mathbf{v}_E + \overbrace{\mathbf{v}_p}^{\propto \text{ i}-\text{inertia}} = -\frac{\boldsymbol{\nabla}_\perp \phi \times \mathbf{B}_0}{B_0^2} + \frac{1}{\omega_{c,i} B_0}\left[-\frac{\partial \boldsymbol{\nabla}_\perp \phi}{\partial t} - (\mathbf{v}_E \cdot \boldsymbol{\nabla}_\perp)\boldsymbol{\nabla}_\perp \phi \right] \; .$$

- *Ion continuity*:

$$\frac{\partial n_i}{\partial t} + \boldsymbol{\nabla}_\perp \left[n_0 (\mathbf{v}_E + \mathbf{v}_p) \right] = 0 \; .$$

- *Massless electrons*, $m_e \ll m_i$:

$$\frac{\tilde{n}_e}{n_0} = \frac{e\tilde{\phi}}{T_e} \; .$$

- *Quasi-neutrality*: to first order we have $\tilde{n}_i = \tilde{n}_e$.

- **NL drift-wave (Hasegawa-Mima) equation:**

$$\frac{\partial}{\partial t}\left(\nabla^2\tilde{\phi} - \tilde{\phi}\right) - \left[\boldsymbol{\nabla}\tilde{\phi}\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\right]\nabla^2\tilde{\phi} - v_{D,e}\frac{\partial\tilde{\phi}}{\partial y} = 0$$

or

$$\overbrace{\frac{D(\nabla^2\tilde{\phi})}{Dt}}^{\text{ion vorticity evolution}} = \overbrace{\frac{\partial\tilde{\phi}}{\partial t}}^{\text{e}-\text{response}} + \overbrace{v_{D,e}\frac{\partial\tilde{\phi}}{\partial y}}^{\text{inhomogeneity}} = 0$$

The low-frequency ion motion in the perpendicular plane (*ion vorticity evolution*) sets up an ion charge density which is neutralized by electron motion in the parallel direction (*e-response*). Stable drift waves are driven by the perpendicular density gradient (*inhomogeneity*).

Parallel between: 2D NS, NL drift wave, Charney equations

$$\frac{\partial \nabla^2 \psi}{\partial t} + (\boldsymbol{\nabla}\psi \times \hat{\mathbf{z}}) \cdot \boldsymbol{\nabla}\nabla^2\psi = 0 \ ,$$

where $\mathbf{v} = \boldsymbol{\nabla}\psi \times \hat{\mathbf{z}}$, $\omega_z = -\nabla^2\psi$.

$$\frac{\partial}{\partial t}\left(\nabla^2\phi - \phi\right) - \left[\boldsymbol{\nabla}\phi \times \hat{\mathbf{z}} \cdot \boldsymbol{\nabla}\right]\nabla^2\phi - v_{D,e}\frac{\partial \phi}{\partial y} = 0 \ ,$$

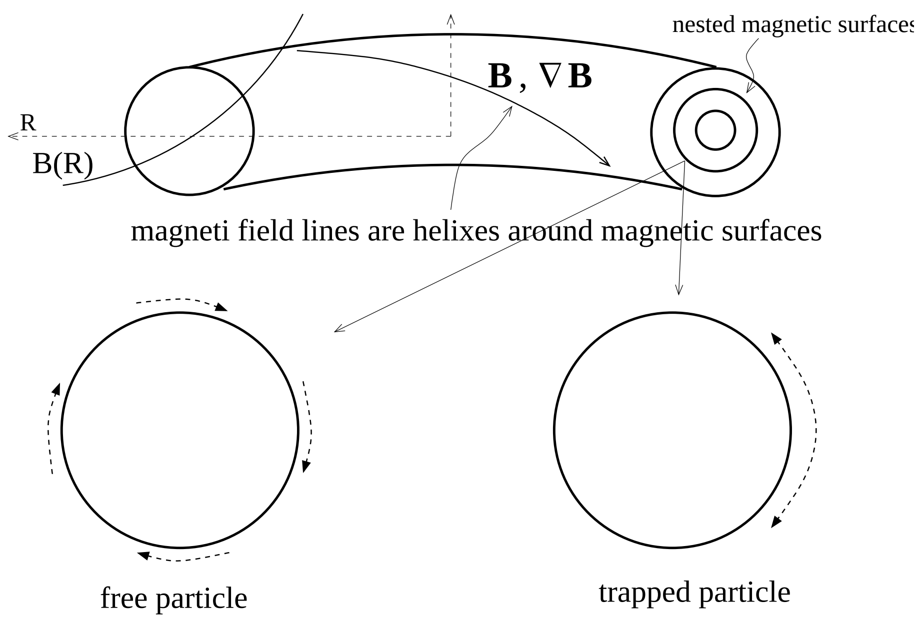
where $\mathbf{E} = -\boldsymbol{\nabla}\phi$, $v_{D,e} \propto dn_0/dx$, and $\omega = k_y v_{D,e}/(1 + k^2\rho_s^2)$.

$$-\frac{\partial}{\partial t}\left(\nabla^2\phi - \phi\right) - \left[\boldsymbol{\nabla}\phi \times \hat{\mathbf{z}} \cdot \boldsymbol{\nabla}\right]\nabla^2\phi - v_R\frac{\partial \phi}{\partial y} = 0 \ ,$$

where $\phi \propto$ fluid depth perturbation, $v_R \propto d(\Omega_z/H)/dx$, and $\omega = k_y v_R/(1+k^2\rho_R^2)$.

ZONAL MODE GENERATION IN TRAPPED ELECTRON MODE TURBULENCE

In a toroidally confined plasmas, part of the electrons are free to follow the magnetic field lines (*free electrons*), while other are forced, by the *parallel gradient in the magnetic field*, to remain within a small volume of the plasma (*trapped electrons*).



- The drag between the trapped and the untrapped population introduces *a phase delay* between $\tilde{n}_e$ and $\tilde{\phi}$, thus driving ***drift waves unstable***.
- Physically, the reduced ability of the trapped part of the electron population to shield out any electrical imbalance due to the perpendicular ion motion *favors instability*.
- The driving free energy still comes from $\nabla n_0 \neq 0$, and the *growth rate is* $\propto \varepsilon_t$*, the trapping fraction*.

Collisionless trapped electron turbulence

The linear evolution of drift waves, their nonlinear interactions, and the trapping/detrapping of electrons in a strongly magnetized plasma lead to *semi-2D electrostatic turbulence*.

A simplified fluid model is given by two *coupled equations for (effective) electron density and ion vorticity (**CTEM model**)*:

$$\frac{\partial n(\mathbf{x},t)}{\partial t} + \nu(n-\phi) + v_d\hat{\alpha}\frac{\partial \phi}{\partial y} = \overbrace{\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}n}^{\text{E}\times\text{B NL}}$$

$$\frac{\partial}{\partial t}\left(1-\boldsymbol{\nabla}^2-\varepsilon_t^{1/2}\right)\phi(\mathbf{x},t) - \varepsilon_t^{1/2}\nu(n-\phi) + v_d(1-\varepsilon_t^{1/2}\hat{\alpha})\frac{\partial\phi}{\partial y} = -\underbrace{\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}^2\phi}_{\text{polarization NL}}$$

Effects of nonlinear terms (according to statistical mechanics)

- **At short $\boldsymbol{\lambda}$:** spectral transfer dominated by polar. NL $\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}^2\phi$ two quadratic invariants: energy and enstrophy $\Rightarrow$ standard *dual cascade* (inverse for energy, direct for enstrophy).
- **At long $\boldsymbol{\lambda}$:** spectral transfer dominated by $E\times B$ NL $\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}n$, one quadratic invariants: energy $\Rightarrow$ *energy to short scales* (production of enstrophy), in a manner that is *not self-similar, anisotropic, and highly nonlocal*
- For long wavelength turbulence

$$\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}^2\phi \ll \boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}n$$

In our study we neglect $\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}^2\phi$ nonlinearity

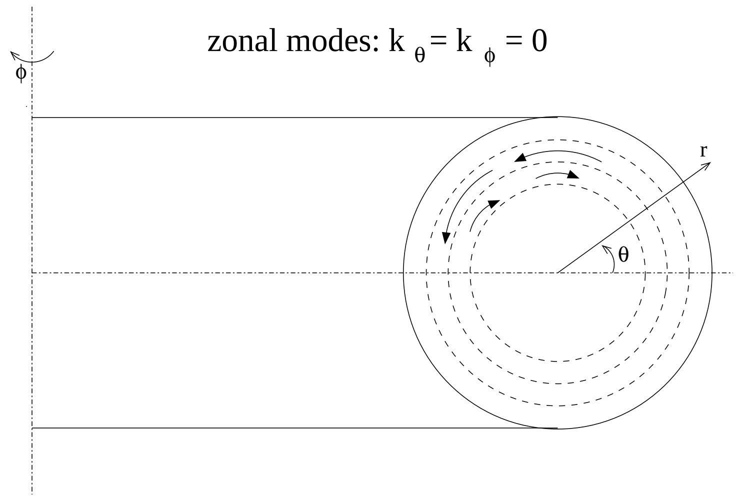
(simulations confirm unimportance of polarization at long wavelengths)

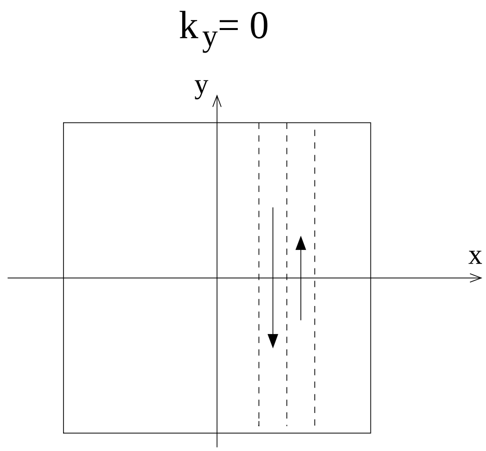
Zonal modes

- In a torus, *zonal modes* are uniform in the poloidal and toroidal direction (poloidal and toroidal wavenumbers equal to zero), but vary radially.

- In 2D magnetized plasma turbulence [slab with $\theta \to y$, $r \to x$, $\phi \to z$], *zonal modes* have $k_y = 0$, $k_x \neq 0$.

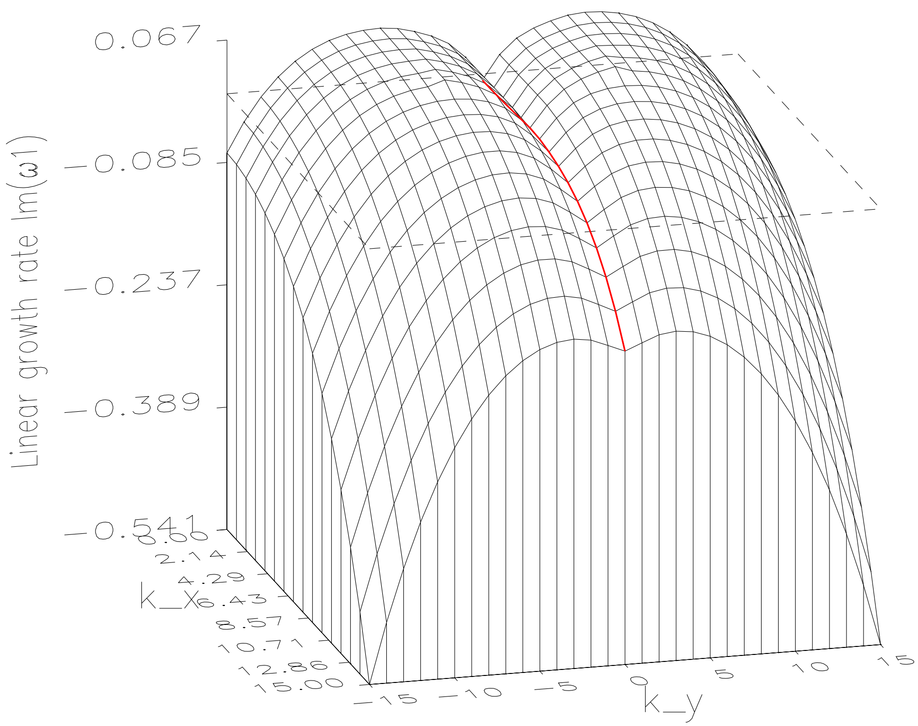
Zonal modes and give rise to a zonally averaged flow.



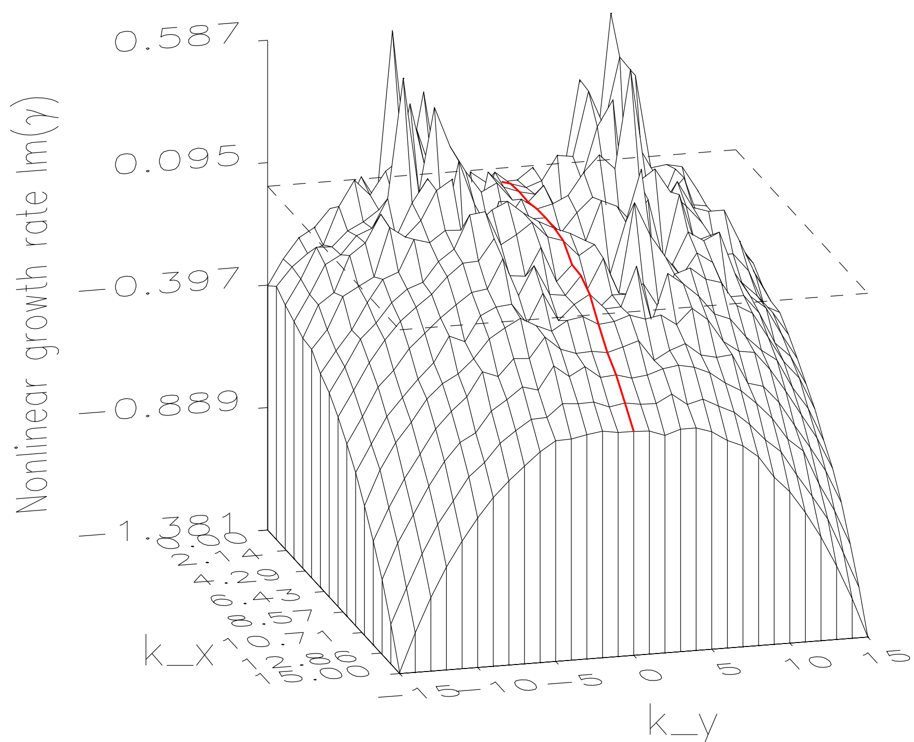


Numerical results

Linear and nonlinear growth rates



Zonal modes (—) are linearly damped

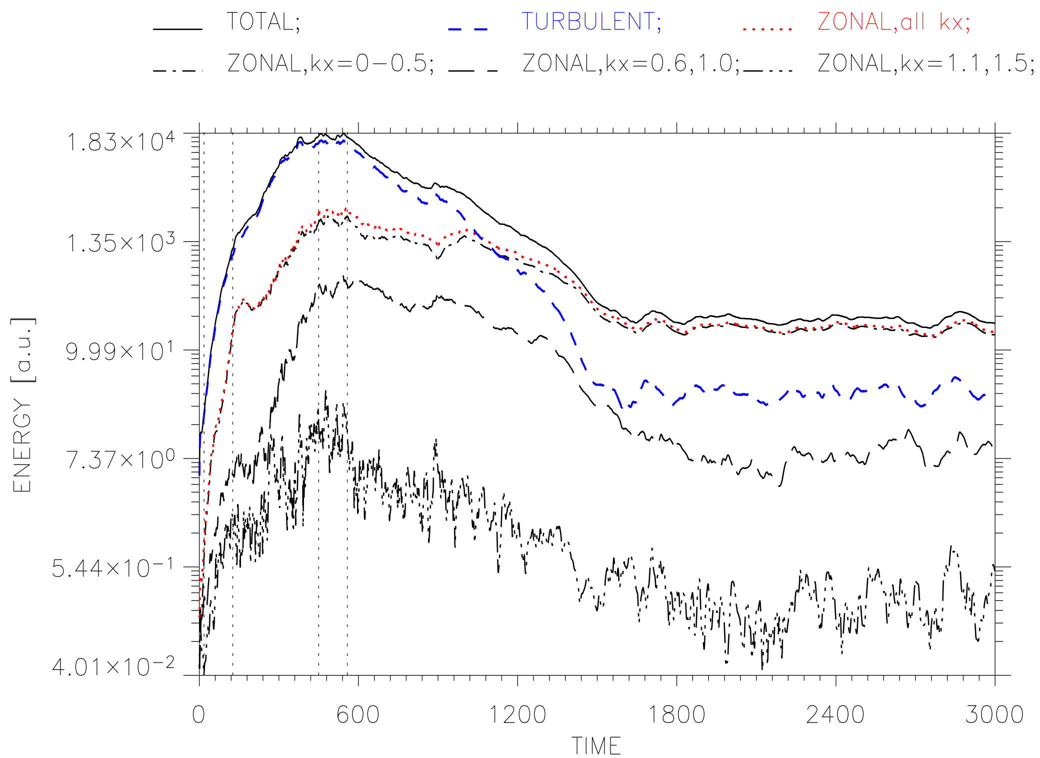


Zonal modes (—) are **strongly nonlinearly damped**

Energy histories

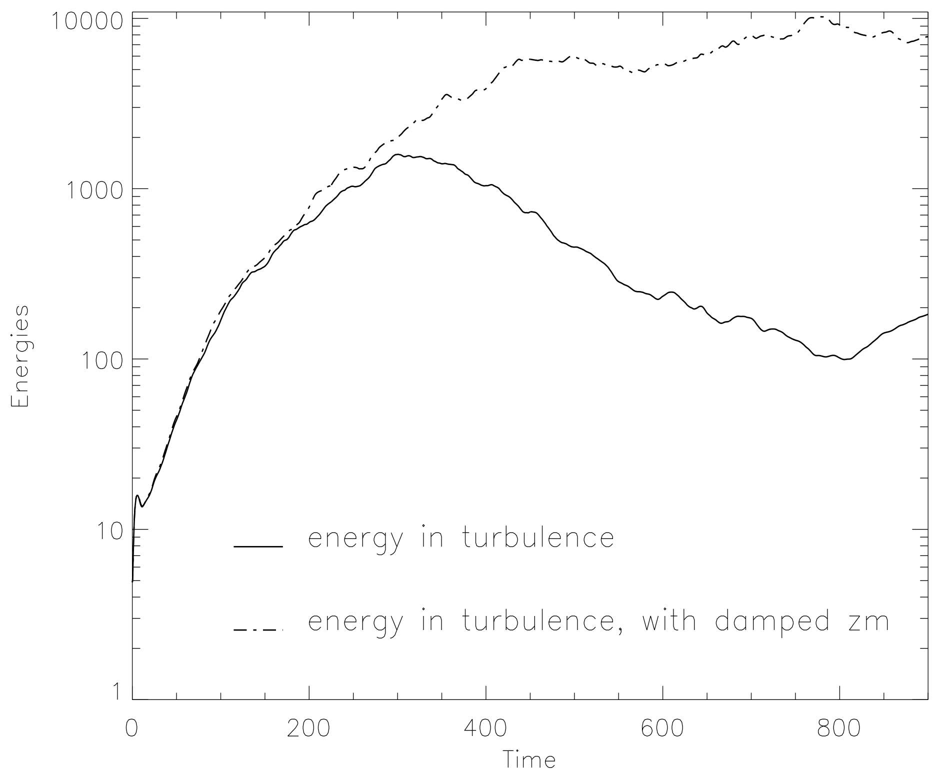
$$\textit{Total energy: } E_{tot} = \sum_{\mathbf{k}} E(\mathbf{k})$$

Zonal energy: $E_z = \sum_{k_x} E(k_x, k_y = 0)$ *Turbulent energy:* $E_t = E_{tot} - E_z$



Zonal modes are **nonlinearly excited**

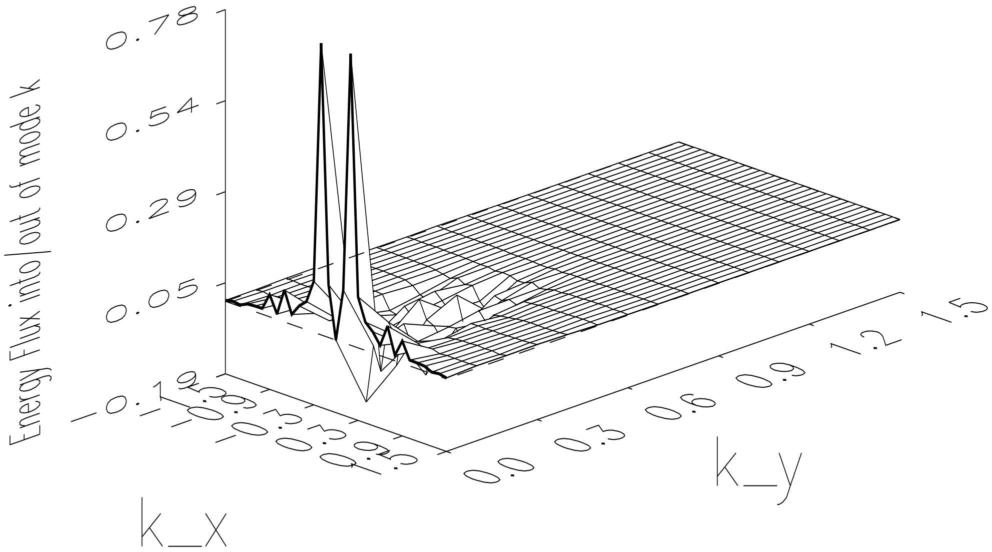
Coupling between turbulent and zonal modes



Energy in turbulence saturates at higher levels when zonal modes are artificially suppressed because ***nonlinear damping of zonal modes is an energy sink for the turbulence***

Spectral energy transfer

Time-averaged energy flux into (positive) or out of (negative) mode **k** due to interactions with all remaining modes



Flux to zonal modes (darker line) is **positive and large** *with respect to the average flux for turbulent modes*

Summary of numerical results

In CTEM, the $E \times B$ nonlinearity, $\nabla\phi\times\hat{\mathbf{z}}\cdot\nabla n$, has two effects:

- *Produce nonlinear eigenmode*, i.e., $n(\mathbf{k}) > \phi(\mathbf{k})$
- *drives zonal flows* (as seen in simulations and experiments)
- *damps zonal flows*: for $k_y = 0$, $\gamma(k_y = 0)E(k_y = 0) < 0$

At saturation, energy is put into the turbulence by linearly and nonlinearly unstable drift-waves, and is transfered to zonal modes to balance their nonlinear damping.

Why nonlinear damping of zonal ($\mathbf{k_y = 0}$) modes?

Energy in mode $\mathbf{k}$,

$$E(\mathbf{k}) = (1 + k^2 - \varepsilon_t^{1/2})|\phi_{\mathbf{k}}|^2 + \varepsilon_t^{1/2}|n_{\mathbf{k}}|^2 \ ,$$

evolves according to

$$\frac{dE(\mathbf{k})}{dt} = 2\gamma_{\mathbf{k}}E(\mathbf{k}) + T(\mathbf{k}) \ ,$$

where T is the nonlinear spectral transfer (conservative: $\sum_{\mathbf{k}} T(\mathbf{k}) = 0$), and $2\gamma E$ is the energy input rate:

$$\gamma_{\mathbf{k}}E(\mathbf{k}) = k_y v_d \hat{\alpha} \varepsilon_t^{1/2} \Im\langle n_{\mathbf{k}}^* \phi_{\mathbf{k}}\rangle - \nu\varepsilon_t^{1/2}|n_{\mathbf{k}} - \phi_{\mathbf{k}}|^2$$

For zonal modes, $\gamma_{\mathbf{k}}E(\mathbf{k}) = -\nu\varepsilon_t^{1/2}|n_{\mathbf{k}} - \phi_{\mathbf{k}}|^2 \leq 0$.

Always damping in the nonlinear regime

$[n(\mathbf{k}) \neq \mathcal{O}(1)\ \phi(\mathbf{k})]$

Why energy transfer to zonal ($\mathbf{k_y = 0}$) modes?

- Saturation regime is *wave-dominated*: $\gamma^{\rm lin} \simeq \gamma^{\rm NL} \ll \omega_D \Rightarrow$ work with equations for *projection coefficients* of the nonlinear solution into the basis set of the linear eigenmodes:

$$\begin{pmatrix} n(k) \\ \phi(k) \end{pmatrix} = \begin{pmatrix} n_1^{\rm lin} & n_2^{\rm lin} \\ \phi_1^{\rm lin} & \phi_2^{\rm lin} \end{pmatrix} \begin{pmatrix} \beta_1(k) \\ \beta_2(k) \end{pmatrix}$$

(similar to the helical decomposition in rotating fluids where the eigenmodes are helicity waves).

- Diagonalized system:

$$\begin{pmatrix} \dot\beta_1(k) \\ \dot\beta_2(k) \end{pmatrix} + \begin{pmatrix} i\omega_1 & 0 \\ 0 & i\omega_2 \end{pmatrix} \begin{pmatrix} \beta_1(k) \\ \beta_2(k) \end{pmatrix} = \frac{1}{R_1(k) - R_2(k)} \begin{pmatrix} b_n \\ -b_n \end{pmatrix}$$

where

$$b_n = -\sum_{\mathbf{k}'} (\mathbf{k}' \times \hat{\mathbf{z}} \cdot \mathbf{k}) \phi(k') n(k-k') \; , \qquad R_j(k) = n_j^{\rm lin}(k)/\phi_j^{\rm lin}(k) \; .$$

- *Anisotropy of spectral transfer* $[\omega_D = \mathcal{O}(1),\, \delta = \nu/\omega_D \ll 1]$

$$\omega_1 \simeq \omega_D + i\delta \xrightarrow{k_y \to 0} 0$$
$$\omega_2 \simeq \delta^2 - i\delta \xrightarrow{k_y \to 0} -i\delta$$

so that

$$\frac{1}{R_1(k) - R_2(k)} \propto \begin{cases} \delta & \text{for } k_y \neq 0 \\ 1 & \text{for } k_y = 0 \end{cases}$$

The rate of energy transfer into zonal modes is larger by a factor $(1/\delta) \gg 1$ than it is for transfer into nonzonal modes

ZONAL MODE GENERATION IN QUASI-GEOSTROPHIC TURBULENCE

Ref.: Chekhlov et al., Physica D 98 (1996) "The effect of small-scale forcing on large-scale structures in 2D flows"

An homogeneous fluid on a rotating sphere obeys the barotropic vorticity equation in the β-plane approx. (Coriolis par. $f = f_0 + \beta y$)

$$\frac{\partial \zeta}{\partial t} + \underbrace{\frac{\partial(\nabla^{-2}\zeta, \zeta)}{\partial(x,y)}}_{\text{NL term}} + \underbrace{\beta \frac{\partial}{\partial x}(\nabla^{-2}\zeta)}_{\text{driving}} = \underbrace{\nu_0 \nabla^2 \zeta}_{\text{dissipation}} + \underbrace{\xi}_{\text{forcing}}$$

- **linear limit w/o forcing**: *planetary (Rossby) waves*

$$\omega = -\beta \, (k_x / k^2)$$

- **NL limit w/ β-term**: classical *isotropic 2D turbulence*
- **Full equation:** ***interaction of 2D turbulence (isotropy) with Rossby waves (anisotropy).***

Asymptotic analysis

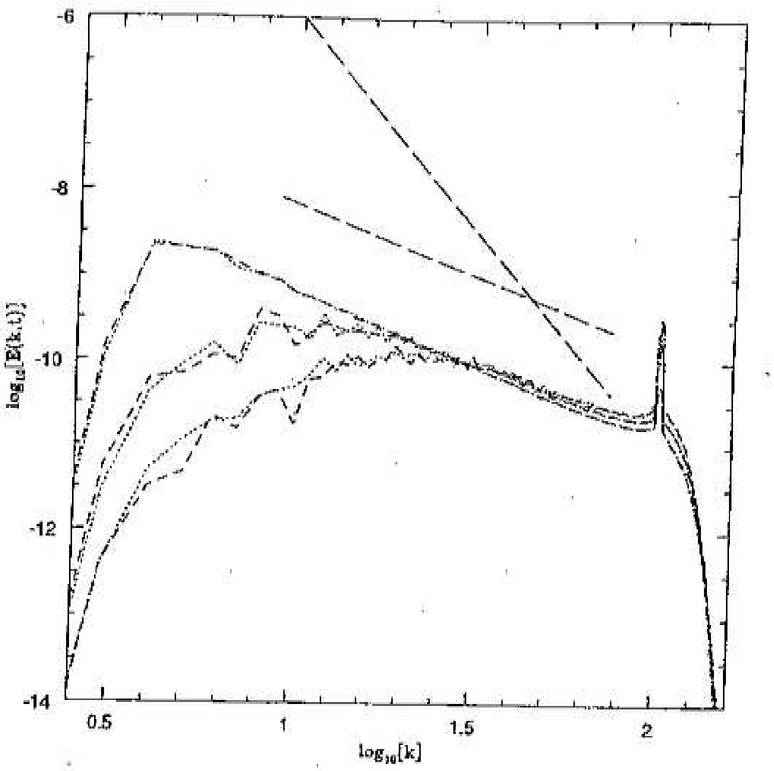
- for $k \to \infty$, β-effect small: isotropic 2D turbulence with

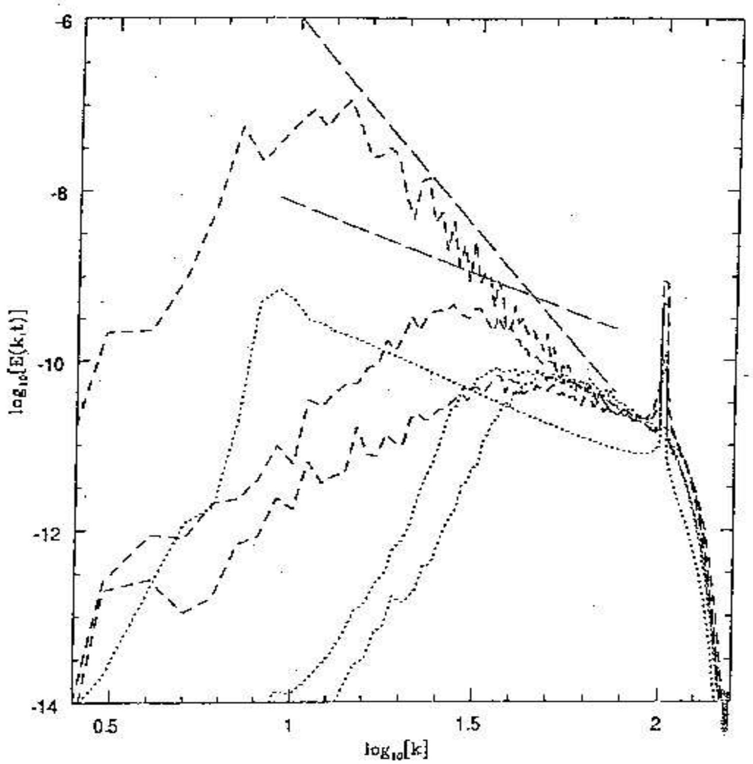
$$E_K(k) = C_K \epsilon^{2/3} k^{-5/3} \quad \text{Kolmogorov}$$

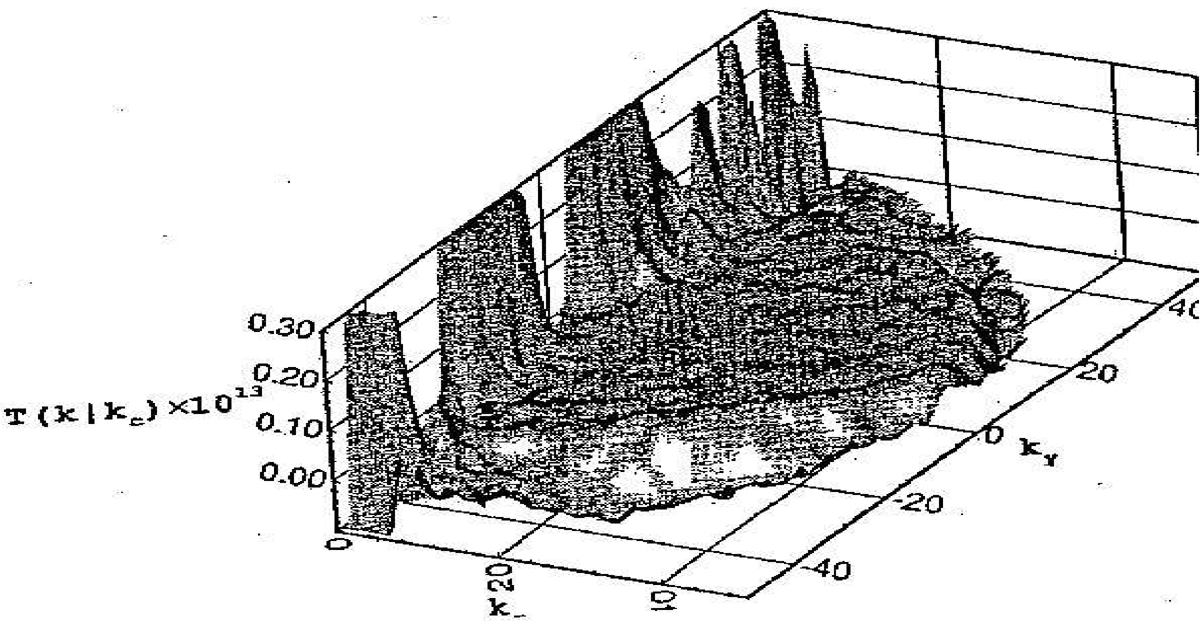
- for $k \to 0$, β-effect relevant: Rossby waves dominate, and

$$E_R(k) = C_R \beta^2 k^{-5} \quad \text{Rhines}$$

Direct Numerical Simulation [$\phi = 0$ $(k_y = 0)$ and $\phi = \pm\pi/2$ $(k_x = 0)$]







Most of the energy is funneled into modes with small k_x corresponding to nearly 1D, zonal structures

CONCLUSIONS

- There is a *broad class of systems exhibiting* ***spectral condensation phenomena***. They are all characterized by:
  - a competition between some type of symmetry breaking (routed in the wave physics) and the isotropy of the nonlinearity, establishing a *preferred direction* $\hat{\mathbf{n}}$,
  - a linear wave frequency which is *minimum* for $\mathbf{k}\cdot\hat{\mathbf{n}} = 0$,
  - a spectral transfer to waves with $\mathbf{k}\cdot\hat{\mathbf{n}} = 0$ which is *strongly enhanced, producing global-scale flows and fields* as a pronounced spectrum feature with the minimum wave frequency.

- *In CTEM turbulence of magnetized plasmas:*
  - $(\boldsymbol{\nabla}\phi\times\hat{\mathbf{z}}\cdot\boldsymbol{\nabla})n$ NL term, both *drives zonal modes to finite amplitudes* and contributes to the *saturation of the turbulence via a nonlinear damping of zonal modes*.
  - The nonlinear coupling between zonal modes ($k_y = 0$) and turbulent ($k_y \neq 0$) modes *reduces the turbulence level at saturation*.

- *In quasi-geostrophic β-plane turbulence:*
  - for $k \to \infty$, the β-effect is small $\Rightarrow$ isotropic 2D turbulence with *Kolmogorov scaling* $[E(k) \propto k^{-5/3}]$
  - for $k \to 0$, the β-effect is relevant $\Rightarrow$ Rossby waves dominate with *Rhines scaling* $[E(k) \propto k^{-5}]$
  - most of the energy is funneled into *small* k_x *zonal structures*