# A stable static Universe?

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Archive number: gr-qc/0405105

Lammi, Finland 17-22 August, 2004

# Summary

- Introduction to Einstein's Universe
- Eddington's instability analysis
- New hypothesis based on emergent gravity
- Free energy based analysis
- Additional remarks

# A bit of history on the Einstein Universe

• In 1917 Einstein proposed that the Universe could be, on the overall, a three-dimensional sphere with no other evolution than that provided by local physics. He arrived to this proposal based on two main ideas:

1. There should not be inertia relative to space.

2. The relative velocities of the stars are very small in comparison with the speed of light.

To make this sort of equilibrium state for the Universe compatible with his geometrical field equations, he introduced the afterward famous cosmological constant $\Lambda$.

• The troubles with this model began this same year, when de Sitter showed that starting with a cosmological constant one can construct a non-trivial cosmological model with no matter whatsoever.

• Weyl in 1923 and Eddington in 1924 pointed out that the already observed redshift of spiral nebulae could be accommodated in de Sitter's model.

• In 1930, Eddington proved that Einstein's static Universe was unstable under homogeneous and isotropic departures from the equilibrium state. The Einstein model started to be considered as a possible initial state for the Universe that once destabilized would start to expand.

Eddington did not clearly analyze what could trigger the development of the instability, but vaguely associate it with the formation of condensations.

• Nowadays we know that the instability of models of the Universe that are closed, and homogeneous, isotropic and static on the overall, is a subtle issue:

1. If the equation of state of matter is such that its associated speed of sound $c_s$ is greater than $1/\sqrt{5}$ all the physical inhomogeneous perturbations are neutrally stable. The Jeans scale for the formation of condensations is a significant fraction of the maximum attainable scale. Therefore, for higher enough speeds of sound, only the Universe as a whole could develop an instability.

2. In order to really depart from the Einstein state one would need a global decrease in pressure in the entire Universe, at least in the simplest case in which matter satisfies an equation of state. This suggests that a static Universe describable in the cosmological scales as filled with dust, for which $p = 0$, could not be able to change its static global state, but only develop instabilities on smaller scales.

On the other extreme, a static Universe filled with radiation could in principle exit from this state towards a Friedman expansion by decreasing its pressure. Here we will concentrate on this later model and its instability.

# Eddington's instability analysis

Let us consider a metric

$$ds^2 = -N^2 dt^2 + a^2(t)\Omega_{ij} dx^i dx^j,$$

where $N$ is the lapse, $a(t)$ the scale factor and $\Omega_{ij}$ the metric on the unit three sphere.

The total gravitational-matter action can be written as

$$S = \boxed{\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda) - \int d^3x\sqrt{-g}\,K}$$

$$\boxed{+\int d^4x\sqrt{-g}\,p(g_{\mu\nu}).}$$

For our particular metric ansatz it reduces to

$$S = \boxed{\frac{2\pi^2}{8\pi G}\int dt\; Na^3\left[3\left(-\frac{\dot{a}^2}{a^2}\frac{1}{N^2} + \frac{1}{a^2}\right) - \Lambda\right]}$$

$$\boxed{+2\pi^2\int dt\; Na^3\,p(N).}$$

The specific form of $p(N)$ depends on the matter equation of state $p = p(\rho)$ through the condition

$$N\frac{\partial p}{\partial N} = -(\rho + p).$$

For example, for radiation $\rho = 3p$ the previous condition yields $p = CN^{-4}$.

By looking at the previous action and having in mind that we are interested on the analysis of the static solutions of the system, we can define a different and simpler functional containing all the relevant information:

$$L_{\text{st}} = Na^3 \left[\frac{3}{a^2} - \Lambda + \tilde{p}(N)\right].$$

We will denote by $\tilde{p} = 8\pi G p$ and $\tilde{\rho} = 8\pi G \rho$, rescaled energy density and pressure.

We can easily see that by varying $L_{\text{st}}$ with respect to $N$ and $a$ and setting $N = 1$ afterwards we obtain

$$(\Lambda + \tilde{\rho})a^2 = 3$$
$$(\Lambda - \tilde{p})a^2 = 1.$$

In the case of a Universe filled with radiation, $\tilde{\rho} = 3\tilde{p}$, these relations give us the radiation Einstein conditions

$$\tilde{\rho} = \Lambda, \qquad a_0^2 = (3/2)\Lambda^{-1}.$$

By looking at the functional $L_{\mathsf{st}}$, (setting $N = 1$), one can also see that the Einstein point is not stable. Taking into account that the kinetic term for the scale factor enters the gravitational action with a negative sign, the local maxima of the functional $L_{\mathsf{st}}$ will correspond to unstable points. This is just the case for the Einstein point (one can perform explicitly the second variation with respect to $a$ to check this local behaviour). This is essentially Eddington's instability.

# Emergent gravity

- General relativity is commonly considered to be a low-energy effective theory that emerges from a deeper underlying structure. A particular realization of this situation is suggested by the gravitational features showing up in many condensed matter system (such as liquid Helium) in the low-energy corner. These type of systems suggest that both, matter particles and interaction fields, could be different emergent features of the underlying system: They will correspond to quasiparticles and collective-field excitations of a multi-particle quantum system. For example, in the phase A of $^3$He, the quasiparticles correspond to Weyl fermions and the collective fields to electromagnetic and gravitational (geometrical) fields.

• Imagine now that a Universe of the Einstein type was the effective result of describing the geometric and matter-like degrees of freedom emerging from the underlying structure. A photons-filled Einstein Universe will have a specific temperature. In the standard general relativity, the stability of the system is analyzed under the assumption of adiabaticity: There is no heat transfer in or out the Universe because there is no "outside the Universe".

• However, in the emergent picture described above there is not any a priory reason to consider the system as effectively closed (let us remark that this is a non standard general relativistic behaviour). Therefore, it is natural to ask what would happen when perturbing the Einstein state if the temperature of the underlying structure stays constant. We will not enter on what sets and controls this temperature, but only assume that it is independent from the behaviour of the effective Universe.

# Free-energy-based analysis

Let us now take a completely different point of view. Let us analyze what happen when the perturbation to the Einstein model is performed as immersed in a thermal reservoir at a fixed temperature. For that let us consider the free energy of static gravitational configurations. The free energy of the purely gravitational part of a static configuration is determined by the Euclidean action of the configuration:

$$\boxed{F_0 = -\frac{1}{16\pi G}\int d^3x\sqrt{g_e}\,(R_e - 2\Lambda)}\,.$$

Here, the symbols $R_e$ and $g_e$ stand respectively for the Euclidean curvature and Euclidean metric of the configuration. We are assuming that a proper Einstein-Hilbert behaviour is emerging in the low-energy corner. Let us remain you that this is not what normally happen in the standard condensed matter systems we know of. In these cases the Einstein-Hilbert behaviour is supplemented with non-covariant terms.

Let us now consider the free energy of a gas of photons (radiation) inside a curved but static geometry. The leading term in the temperature on the free energy function is

$$\boxed{F_1 = -\tfrac{\sigma}{3}\int d^3x\sqrt{g_e}\,T^4(x)},$$

where $\sigma \equiv \pi^2 k_B^4/15\hbar^3c^2$ is the Stefan-Boltzmann constant, and

$$T(x) = \frac{T_0}{\sqrt{g_{00}(x)}}$$

the Tolman temperature; (we will see later that there are other contributions to the free energy in lower powers of the temperature). For the particular geometries we are interested in here, the total free energy can be written as

$$F(a,N,T) = F_0 + F_1 = \frac{2\pi^2a^3}{8\pi G}\left[-\frac{3N}{a^2} + N\Lambda - \frac{\tilde{p}}{N^3}\right],$$

with

$$\tilde{p} = \frac{1}{3}\tilde{\rho} = \frac{8\pi G}{3}\sigma T_0^4 : \text{constant.}$$

From this free energy, associated with a static geometry filled with radiation at a temperature $T_0$,

we can obtain the Einstein static condition. It corresponds to the one that extremises the function $F$. Variation with respect to $N$ with an afterwards evaluation in $N = 1$ gives

$$(\Lambda + 3\tilde{p})a^2 = (\Lambda + \tilde{\rho})a^2 = 3.$$

Variation with respect to $a$ yields

$$(\Lambda - \tilde{p})a^2 = 1.$$

Therefore we have found the same expressions that before: The conditions for a static Einstein Universe filled with radiation.

By inspection of the free energy function we can see that now the Einstein static point is located at a local minimum. This is the main point we want to highlight in this work.

*If the perturbation of the radiation filled Einstein Universe were done under the influence of an externally fixed temperature, (something outside the realm of standard general relativity) then, the Einstein point would be stable*.

# Intuitive explanation

It is not difficult to understand why this is the case. The Eddington instability of the Einstein state is based on the following fact. The contractive tendency of matter operates strongly in short scales. Instead, the expansive tendency of the cosmological constant operates strongly in large scales. In the Einstein point these two tendencies are exactly balanced. However, if the universe is suddenly made larger, the cosmological constant effect takes over and expand further the Universe. Reciprocally, if the Universe is made smaller the matter dominates and makes the Universe to further contract. However, in the case analyzed here, a sudden expansion of the Universe will be accompanied by the introduction of more photons in the system in order to keep the temperature constant in the now larger volume. This increase on the amount of matter completely counterbalance the cosmological constant tendency making the Universe to contract back to its initial state.

# Curvature corrections

The free energy does contain additional contributions in smaller powers of the temperature. In the high-temperature limit $T^2 \gg \hbar^2 R_e$, the total free energy for a gas of photons in a static spacetime is [11,13]

$$F = \boxed{-\frac{1}{16\pi G}\int d^3x\sqrt{g_e}\,(R_e - 2\Lambda)}$$

$$\boxed{-\frac{\sigma}{3}\int d^3x\sqrt{g_e}T^4 + \bar{\sigma}\int d^3x\sqrt{g_e}T^2[R_e + 6\omega^2]}$$

with

$$\bar{\sigma} = \frac{k_B^2}{36\hbar}, \qquad \text{and} \qquad \omega_\mu = \frac{1}{2}\partial_\mu \ln|g_{00}(r)|.$$

In our particular case, this free energy yields

$$F(N,a,T) = \frac{2\pi^2 a^3}{8\pi G}\left[-\frac{3N}{a^2} + N\Lambda - \frac{\tilde{p}}{N^3} + 8\pi G\bar{\sigma}\frac{6T_0^2}{Na^2}\right],$$

or expressing everything in terms of $\tilde{\rho}$ and denoting the constant factor $(8\pi G/\sigma)^{1/2}6\bar{\sigma}$ by the letter $b$,

$$F(N,a,T) = \frac{2\pi^2 a^3}{8\pi G}\left[-\frac{3N}{a^2} + N\Lambda - \frac{\tilde{\rho}}{3N^3} + b\,\frac{\tilde{\rho}^{1/2}}{Na^2}\right].$$

Varying with respect to $N$ and $a$ we find now

$$(\Lambda + \tilde{\rho})a^2 - b\tilde{\rho}^{1/2} = 3,$$
$$\left(\Lambda - \frac{1}{3}\tilde{\rho}\right)a^2 + \frac{1}{3}b\tilde{\rho}^{1/2} = 1.$$

Manipulating these two conditions one obtains modified Einstein conditions

$$\Lambda = \frac{3}{2a^2} = \frac{\tilde{\rho}}{1 + \frac{2}{3}b\tilde{\rho}^{1/2}}.$$

## Consistency check

Consider the following self consistent procedure:

Calculate first $\langle \psi_T | T_{\mu\nu} | \psi_T \rangle$ on an 3-sphere of arbitrary radius $a$.

Then plug-in the result in the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \langle \psi_T | T_{\mu\nu} | \psi_T \rangle,$$

for metrics of the type $ds^2 = -dt^2 + a^2 d\Omega_3^2$.

From here we can find the radius of the Universe as a function of the temperature. The result found is the same that we have found by varying the free-energy.

# Anisotropic instability

• In standard general relativity the Einstein point for a Universe filled with radiation is also unstable against homogeneous but anisotropic perturbations of the metric of the Bianchi type IX.

Let us calculate the free energy of static configurations of the Bianchi IX type. The general metric for these models is

$$ds^2 = -N dt^2 + \sum_{n=1}^{3} a_n^2 \sigma_n^2,$$
$$\sigma_1 = \sin\psi d\theta - \cos\psi \sin\theta d\varphi,$$
$$\sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi,$$
$$\sigma_3 = -(d\psi + \cos\theta d\varphi).$$

Now, the free energy for these configurations results

$$F(a_1, a_2, a_3, N, T) = \frac{2\pi^2 a_1 a_2 a_3}{8\pi G} \times$$
$$\times \left[ \frac{N a_1^2}{a_2^2 a_3^2} + \frac{N a_2^2}{a_1^2 a_3^2} + \frac{N a_3^2}{a_1^2 a_2^2} - \frac{2N}{a_1^2} - \frac{2N}{a_2^2} - \frac{2N}{a_3^2} + N\Lambda - \frac{\tilde{p}}{N^3} \right].$$

Again, it is not difficult to see that the Einstein point is an extremum of this free energy $a_1^2 = a_2^2 = a_3^2 = (3/2)\Lambda^{-1} = (3/2)\tilde{\rho}^{-1}$, and that it is a local minimum.

# Adjusting the cosmological constant

- Condensed matter systems provide examples in which the vacuum pressure calculated through low-energy mode expansions yield erroneous results.

For example, in a dilute BEC with repulsive interactions we have

$$p_V = -\frac{1}{2}\int d\mathbf{p}^3 E(\mathbf{p}) < 0; \qquad E(\mathbf{p}) = \sqrt{c^2p^2 + p^4/4m^2}$$

$$p_V = \frac{mc^2 n_0}{2} - \frac{1}{2}\int d\mathbf{p}^3 \left( E(\mathbf{p}) - \frac{p^2}{2m} - mc^2 + \frac{m^3c^4}{p^2} \right) > 0$$

- A system with an stable vacuum has to have

$$p_V = 0.$$

If one then inject some quasiparticles into the system with $p_Q > 0$, the system would have to provide a modified $p_V = -p_Q < 0$ in order to stabilize itself. In our analysis this adjustment is

$$\tilde{p}_V = -\Lambda = -\frac{1}{3}\sigma T^4.$$

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