Fluctuations and Stability of Superfluid Turbulence

at mK Temperatures

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Lammi

Superfluid turbulence (T = 0)

(^{4}He, BEC)

Equation of motion : nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V |\psi|^2 \psi$$

Madelung transformation: $\psi = a e^{i\phi}$

$$\vec{v} = \frac{\hbar}{m} \nabla \phi$$

$$\rho = m a^2$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$ continuity equation

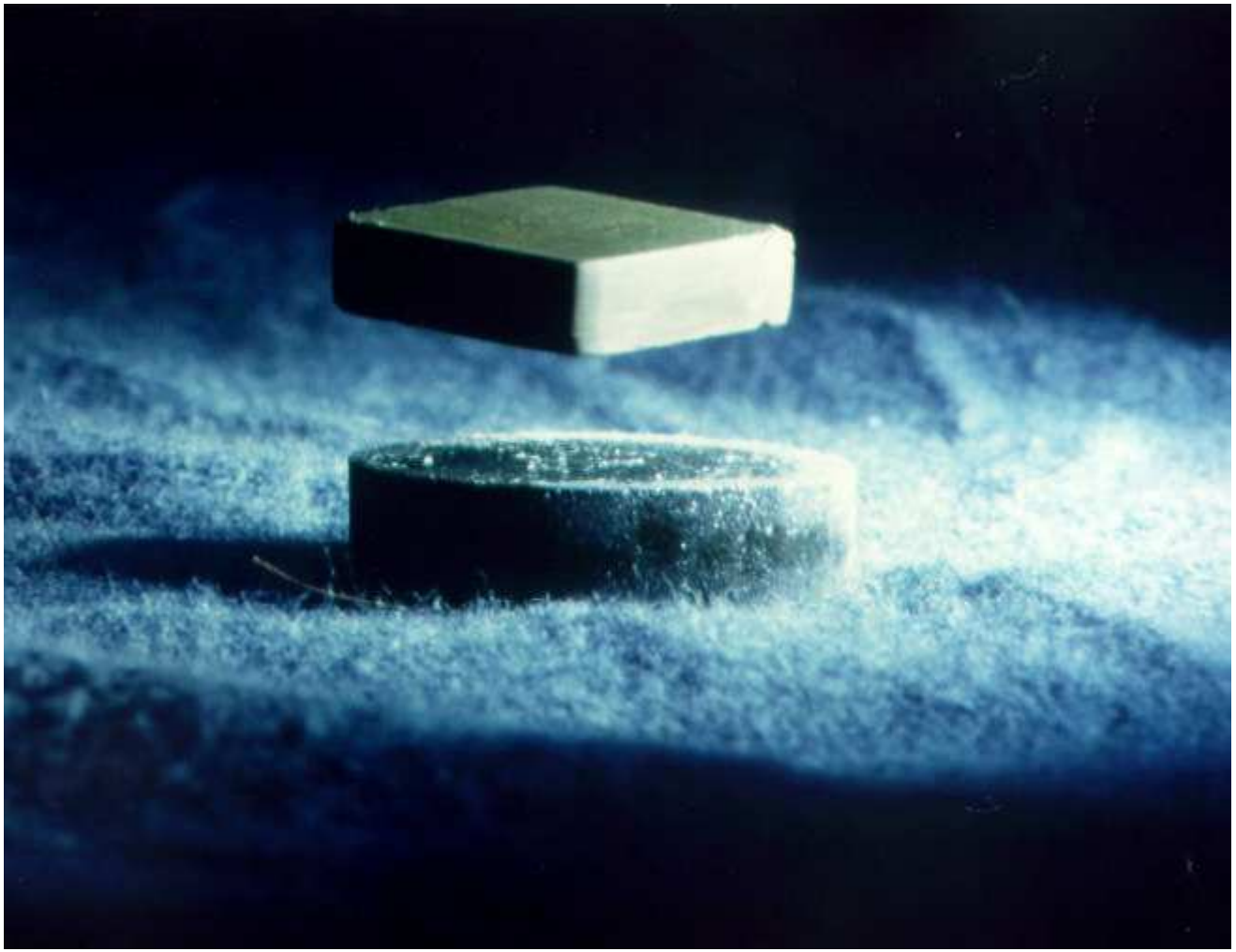
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = -\nabla \frac{V a^2}{m}$$ Euler equation

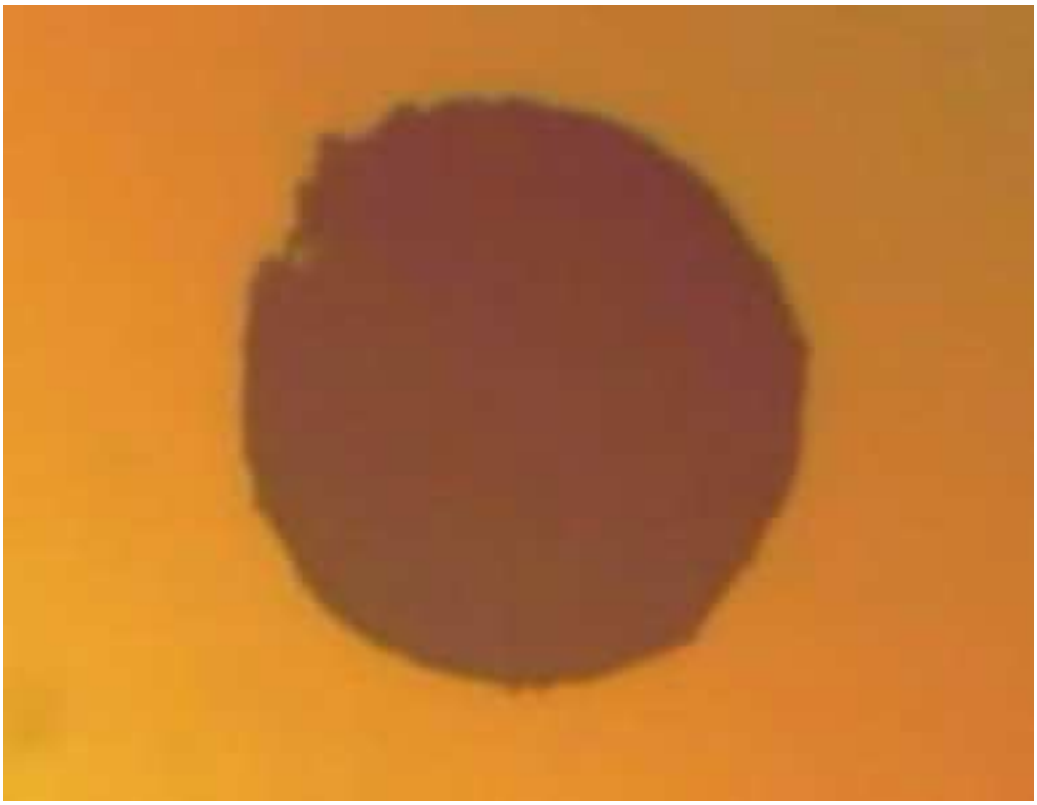
no viscosity, no Reynolds number

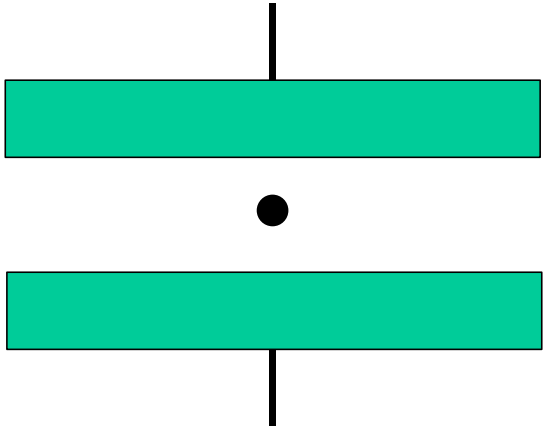
pure potential flow: $\mathrm{rot}\, \vec{v} = 0$

quantized circulation: $\oint \vec{v} \cdot d\vec{r} = N\kappa$, vortices with N = 1

$\boxed{\text{superfluid turbulence = tangle of singly quantized vortices}}$



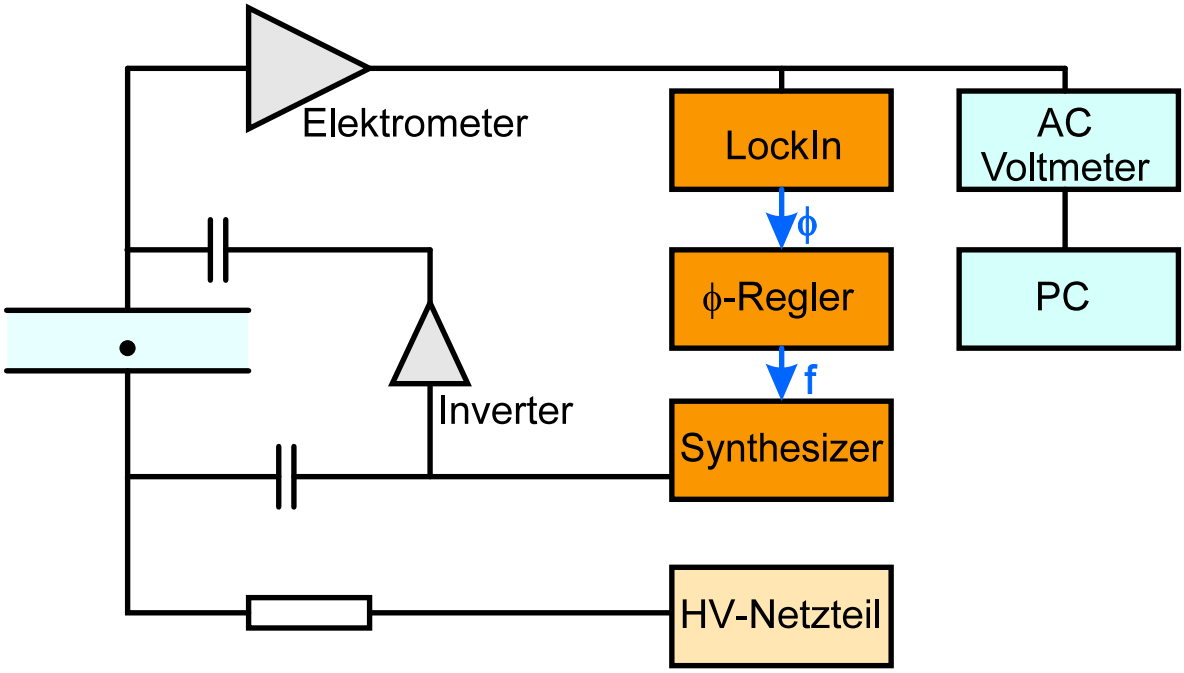


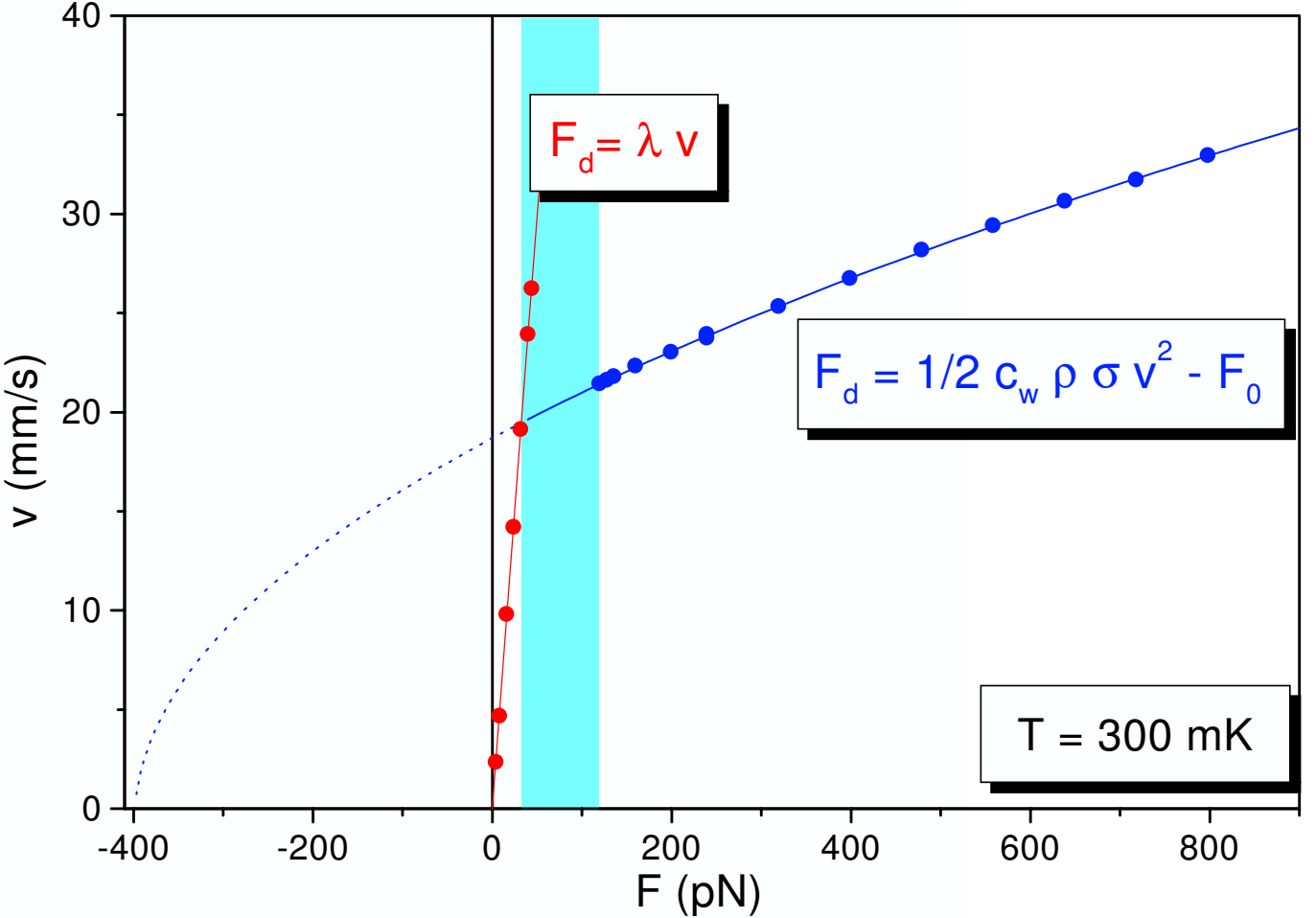


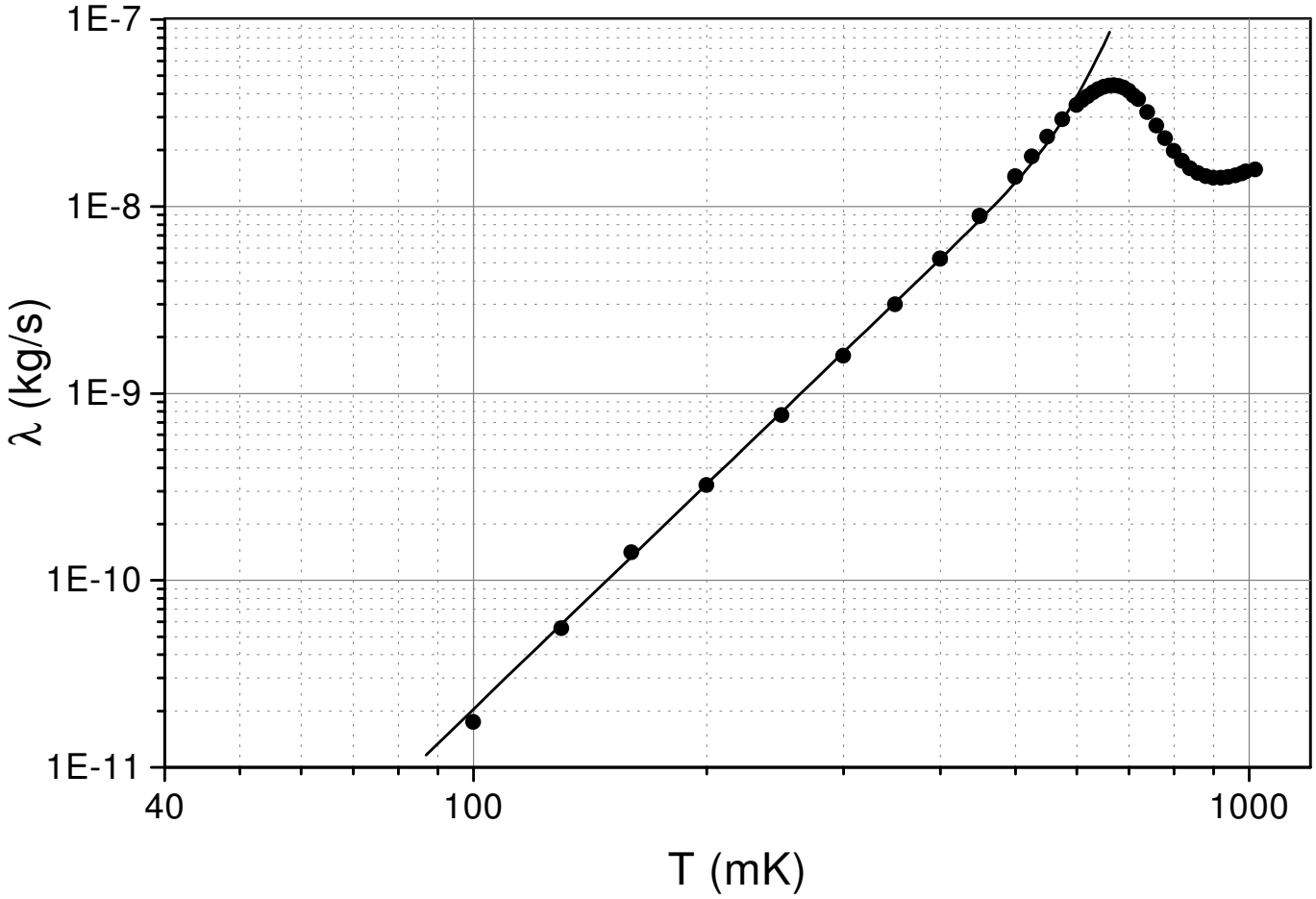
Nb capacitor: spacing 1 mm, diameter 4mm

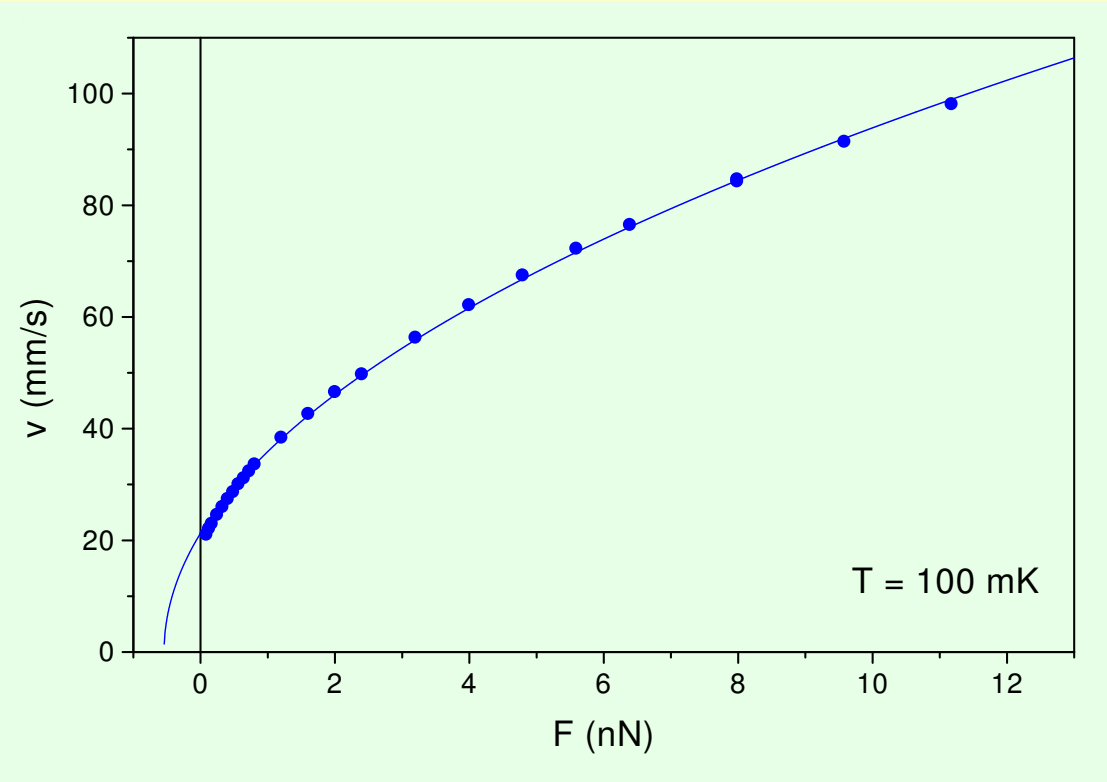
permanent magnet: spherical, radius 0.1 mm,
electric charge ca. 1 pC

driving force: $F = q\, U_{ac} / d$
induced current: $I = q\, v / d$
oscillation amplitude 50 nm to 50 μm


Elektrometer
LockIn
AC
Voltmeter
ϕ
ϕ-Regler
PC
f
Inverter
Synthesizer
HV-Netzteil


$F_d = \lambda v$
$F_d = 1/2\, c_w \rho \sigma v^2 - F_0$
T = 300 mK
v (mm/s)
F (pN)
40
30
20
10
0
-400
-200
0
200
400
600
800

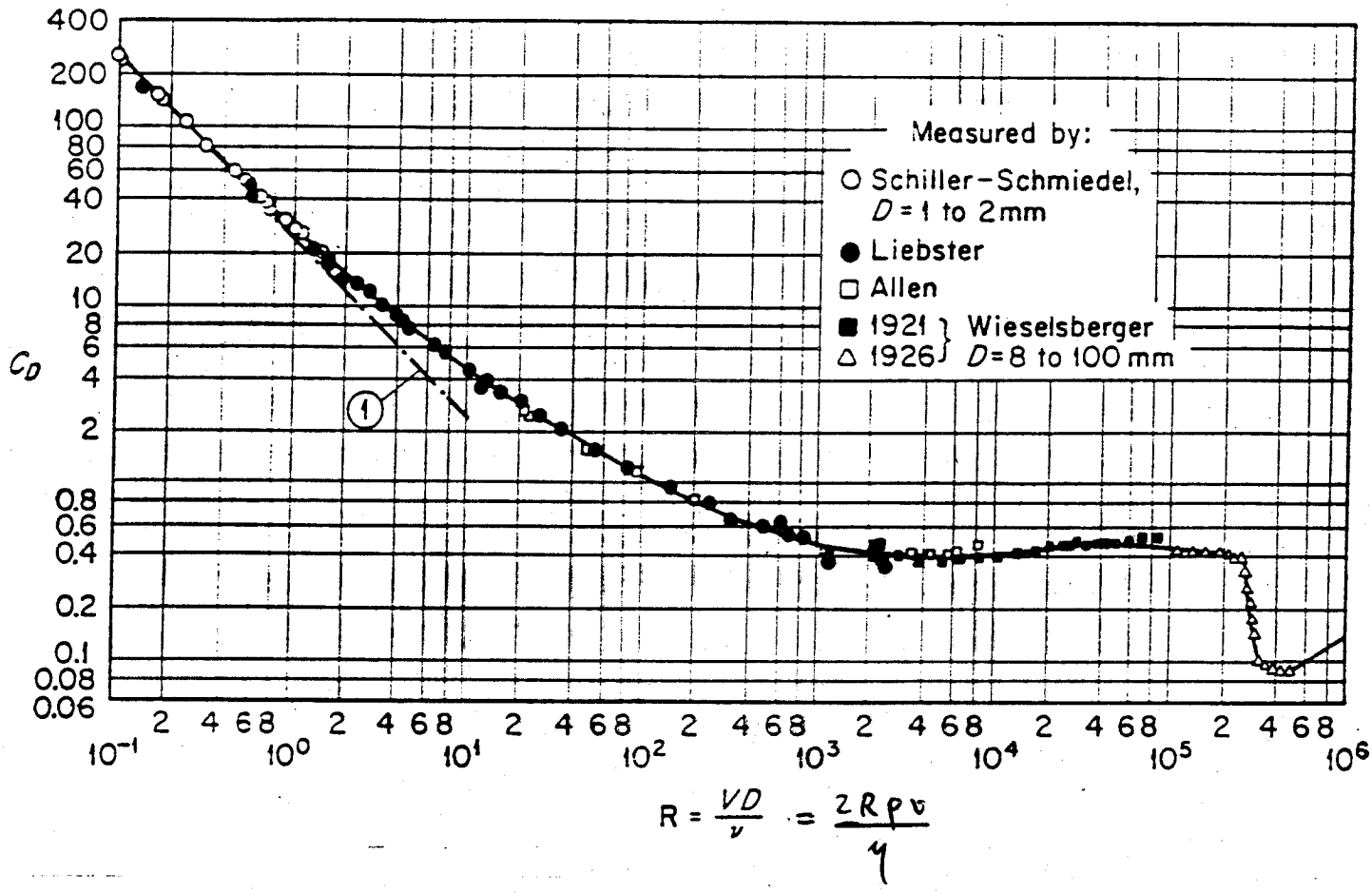

1E-7
1E-8
1E-9
1E-10
1E-11
λ (kg/s)
40
100
1000
T (mK)


v (mm/s)
100
80
60
40
20
0
0
2
4
6
8
10
12
F (nN)
T = 100 mK

SPHERE IN CLASSICAL VISCOUS FLUID

UNIFORM MOTION

(FROM HANDBOOK OF FLUID DYNAMICS)

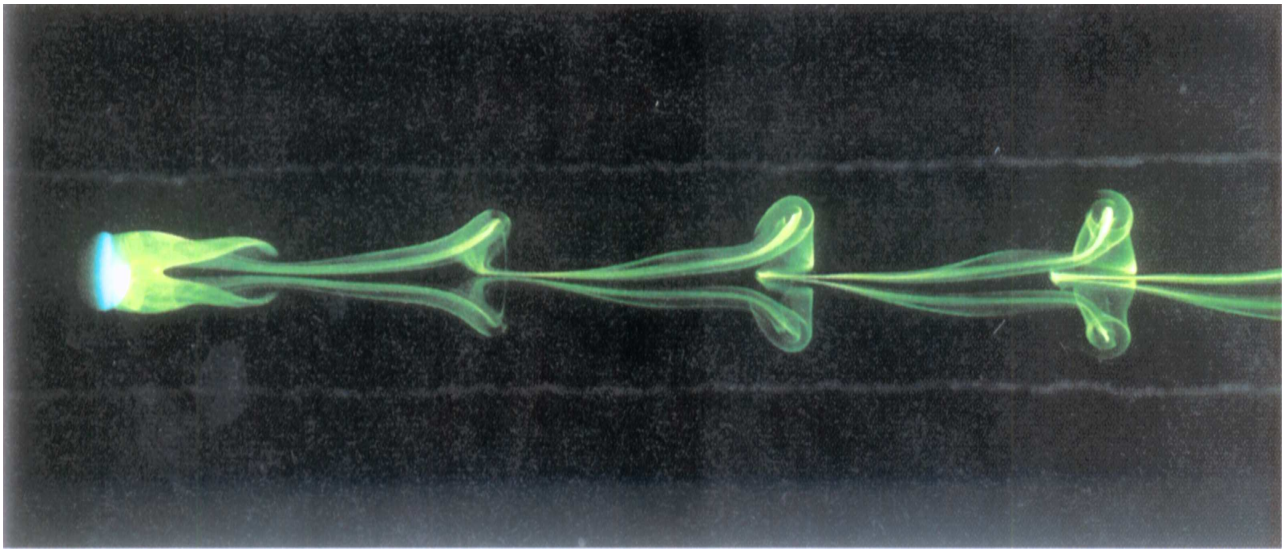


① : laminar $\quad F_f = 6\pi\eta R v$

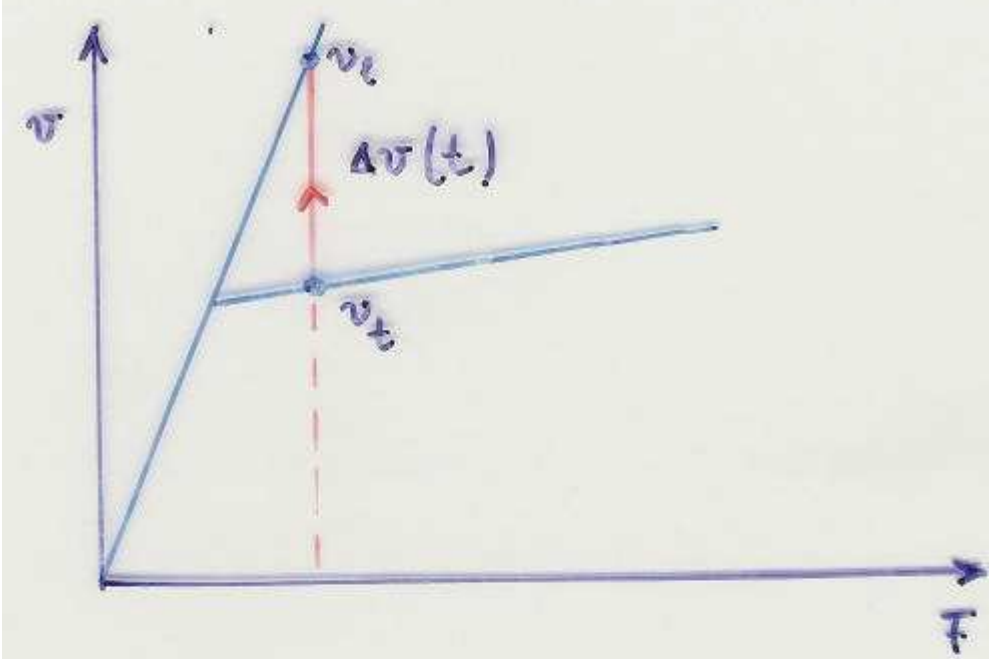
$$c_D \equiv \frac{F_f}{\frac{\rho}{2}\pi R^2 v^2} = \frac{24}{Re} \qquad (Re \ll 1)$$

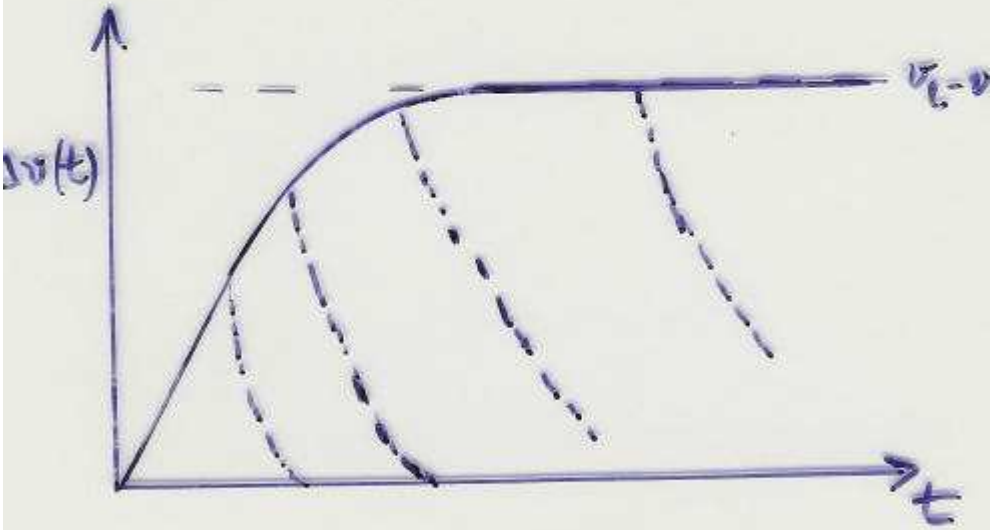
turbulent $\quad F_f = \frac{1}{2} c_D \rho \pi R^2 v^2$

$$c_D \simeq 0.4$$



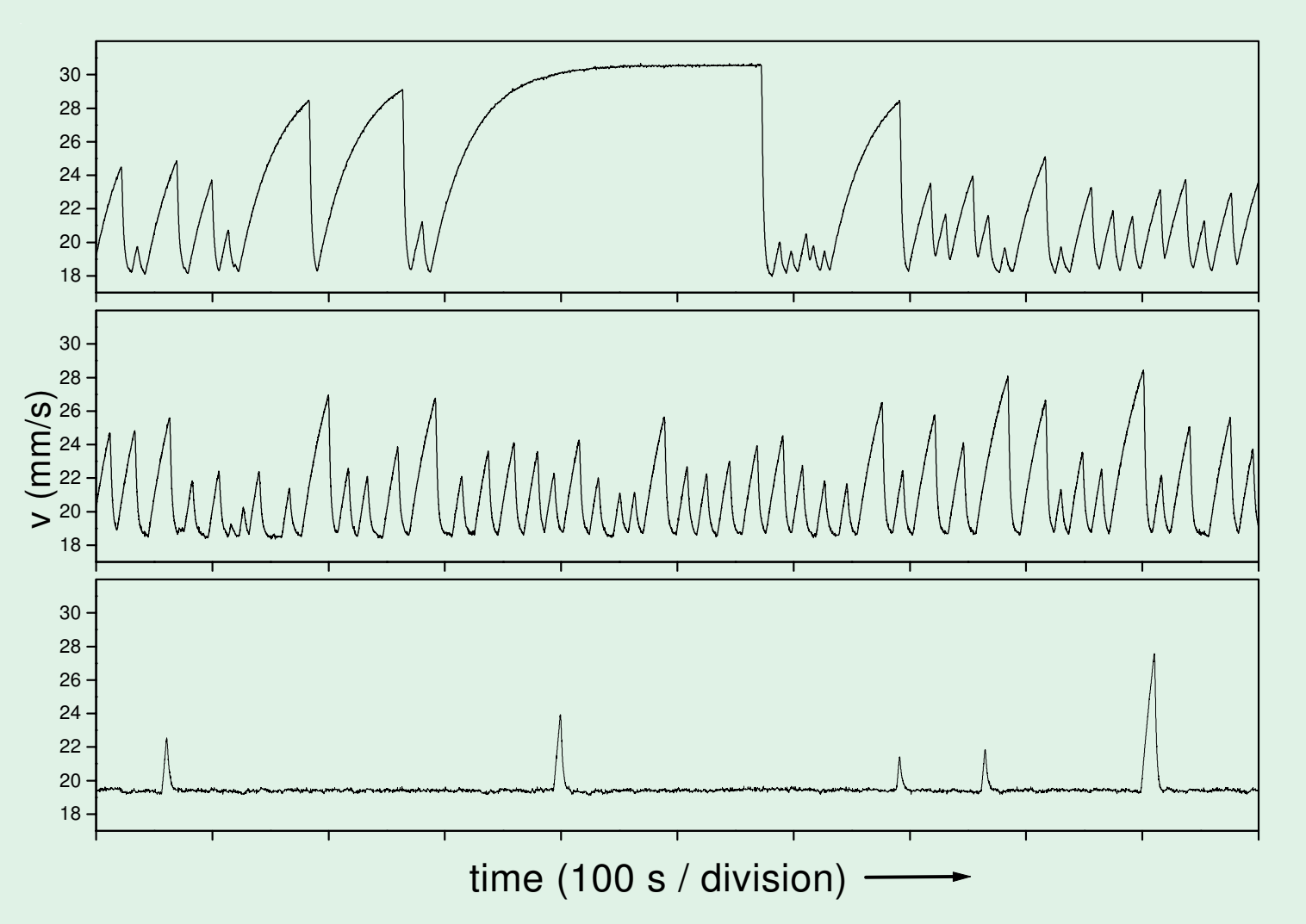
v2.jpg (1488x2616x24b jpeg)



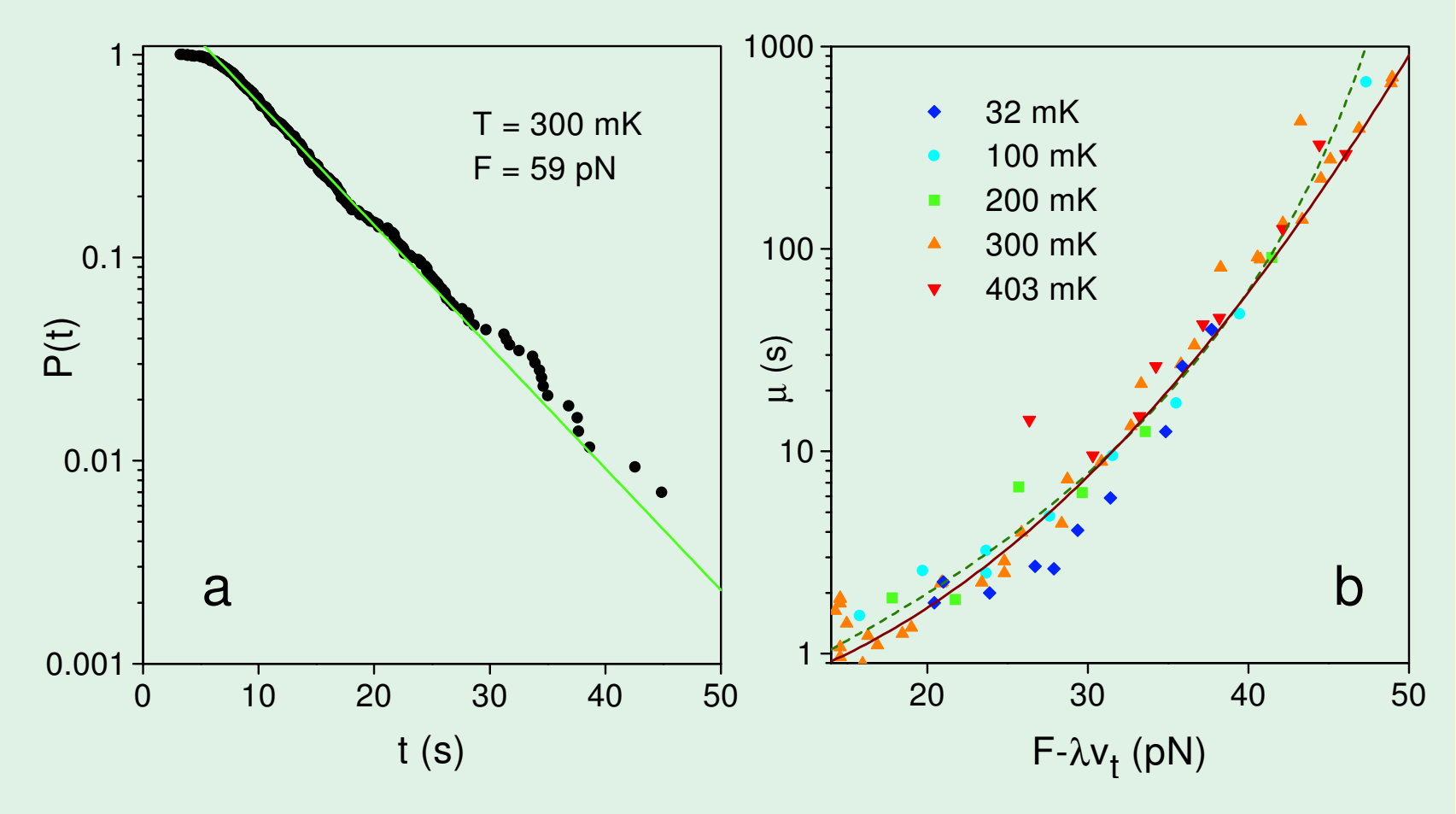


$$\Delta v(t) = (v_\ell - v_t)\left(1 - e^{-t/\tau}\right)$$

Intermittent Switching



Turbulent Statistics



$$\boxed{\langle t \rangle = t_0 \, exp\left(\frac{F_{turb}}{F_0}\right)^2}$$

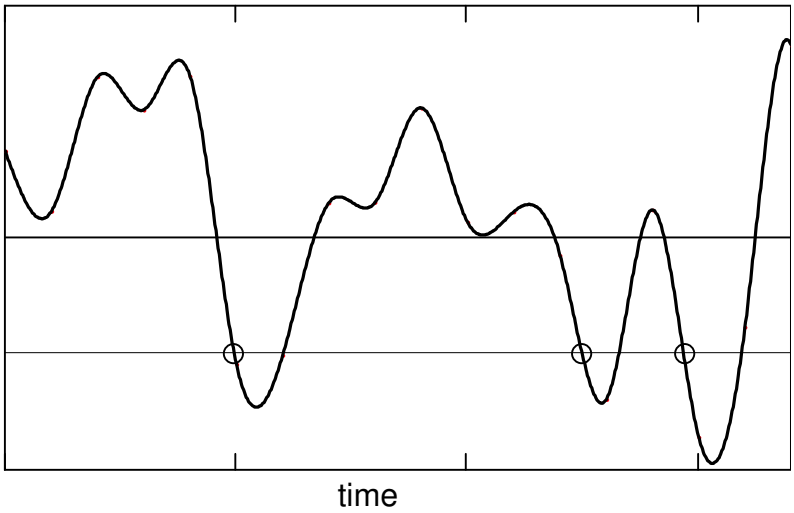
$$t_0 = 0.5\,s$$

$$F_0 = 18\,pN$$

$$\varepsilon = \frac{P_{turb}}{M} \sim \kappa^3 \langle L^2 \rangle$$

$$\frac{P}{P_0} = \frac{\langle L^2 \rangle}{L_0^{\,2}}, \quad L_0^2 = \frac{P_0}{M\kappa^3} = (2.1 \cdot 10^7\ m^{-2})^2$$

$$\boxed{\langle t \rangle = t_0 \, exp\left(\frac{\langle L^2 \rangle}{L_0^{\,2}}\right)^2}$$


time

$$\boxed{\langle N \rangle = \frac{1}{2\pi}\sqrt{-r''(0)}\, exp\left(-\frac{C^2}{2\sigma_\xi^2}\right)}$$

$$-r''(0) = \int_0^\infty \omega^2 S(\omega)\, d\omega$$

$$\langle N \rangle \cdot \langle t \rangle = 1$$

$$\boxed{\langle t \rangle = \frac{1}{\langle N \rangle} = \frac{2\pi}{\sqrt{-r''(0)}}\, exp\left(\frac{C^2}{2\sigma_\xi^2}\right)}$$

$$\boxed{\langle t \rangle = t_0\, exp\left(\frac{\langle L^2 \rangle}{L_0{}^2}\right)^2}$$

Results:

1. The enstrophy L^2 has a **<u>normal distribution</u>** with a standard deviation

$$\sigma_L = \frac{L_0^2}{\sqrt{2}} = 2.9 \cdot 10^{14}\, m^{-4}$$

the typical vortex spacing is $\frac{1}{\sqrt{L_0}} = 0.22\, mm$

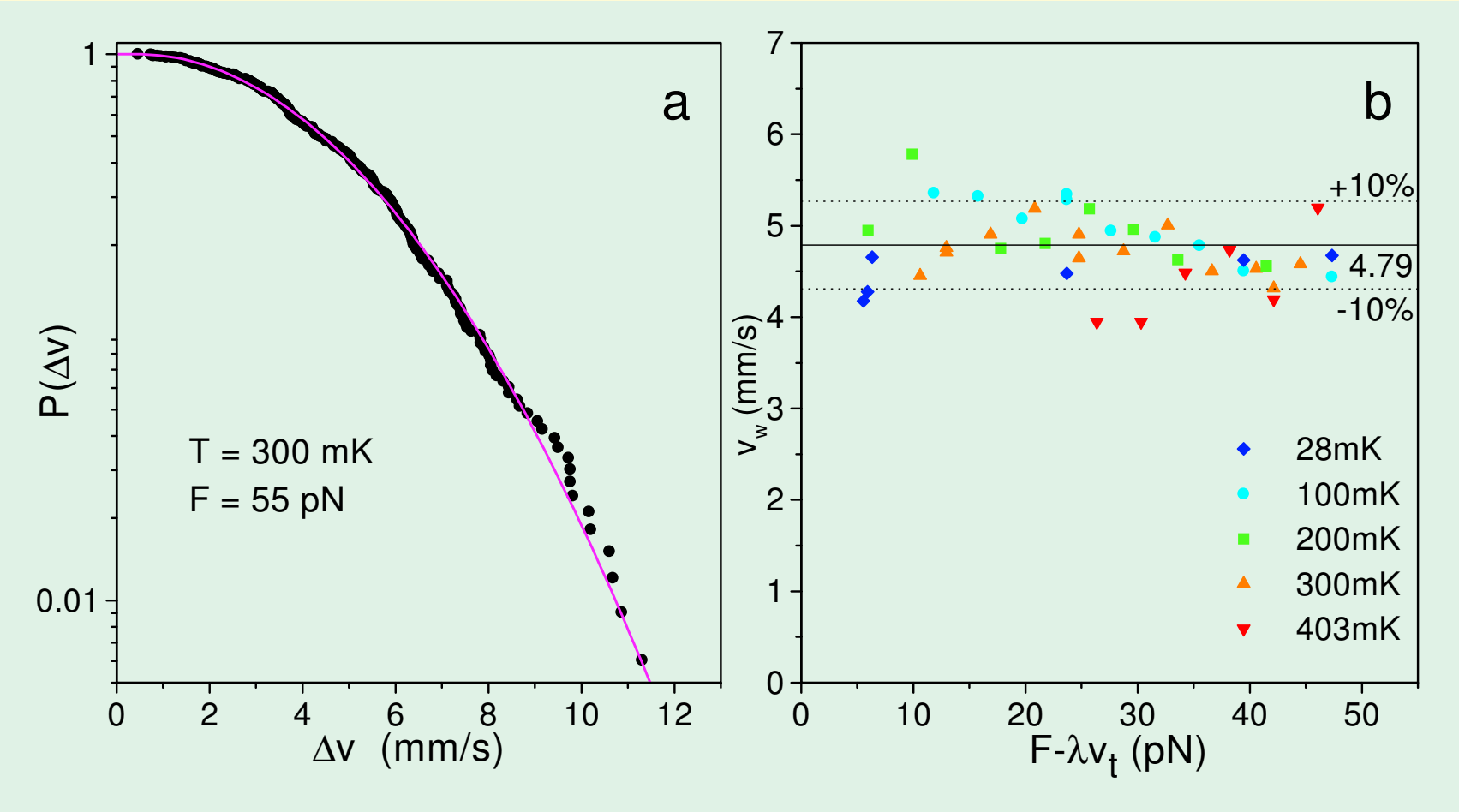
2. The **<u>spectral bandwidth</u>** of the enstrophy fluctuations is

$$\Delta\omega \sim \sqrt{-r''(0)} = \frac{2\pi}{t_0} = 13\ s^{-1}$$

the typical time scale is $\frac{1}{\kappa L_0} = 0.5\, s$

Literature: W. Schoepe, Phys. Rev. Lett. 92, 095301 (2004)

Laminar Statistics



$$R(\Delta v) = \exp(-(\Delta v/v_w)^2)$$

$F(\Delta v) = 1 - \exp(-(\Delta v/v_w)^2)$ "Weibull" distribution

Terminology of elementary statistics

The cumulative distribution function (CDF) $F(x)$

The survival function or reliability function $R(x) = 1 - F(x)$

The probability density function (PDF) $f(x) = dF/dx$

The failure rate or hazard rate $\Lambda(x)$: $dR/dx = -\Lambda(x)R(x)$

$$\Lambda(x) = -d \ln R(x)/dx = f(x)/R(x)$$

$$R(x) = \exp\left(-\int_{-\infty}^{x} \Lambda(x')\, dx'\right)$$

Examples

1) $\Lambda =$ const., $\quad x =$ time

– radioactivity, macroscopic quantum tunneling
– thermal activation over a barrier $\Lambda = \Lambda_0 e^{-E/T}$

$R(t) = e^{-\Lambda t} \qquad F(t) = 1 - e^{-\Lambda t}, \qquad \frac{1}{\Lambda} = \tau$

2) $\Lambda(x) = x$

$R(x) = e^{-\frac{x^2}{2}} \qquad F(x) = 1 - e^{-\frac{x^2}{2}},$ Rayleigh (Weibull) distribution

3) $\Lambda(x) = e^x$

$R(x) = e^{-e^x} \qquad F(x) = 1 - e^{-e^x},$ Gompertz distribution

Lifetime of Laminar Phases

Reliability function: $R(\Delta v) = \exp\left(-\left(\Delta v / v_w\right)^2\right)$.

Failure rate: $\Lambda(\Delta v) = 2\Delta v / v_w^2$.

Transformation of variables from Δv to time t:

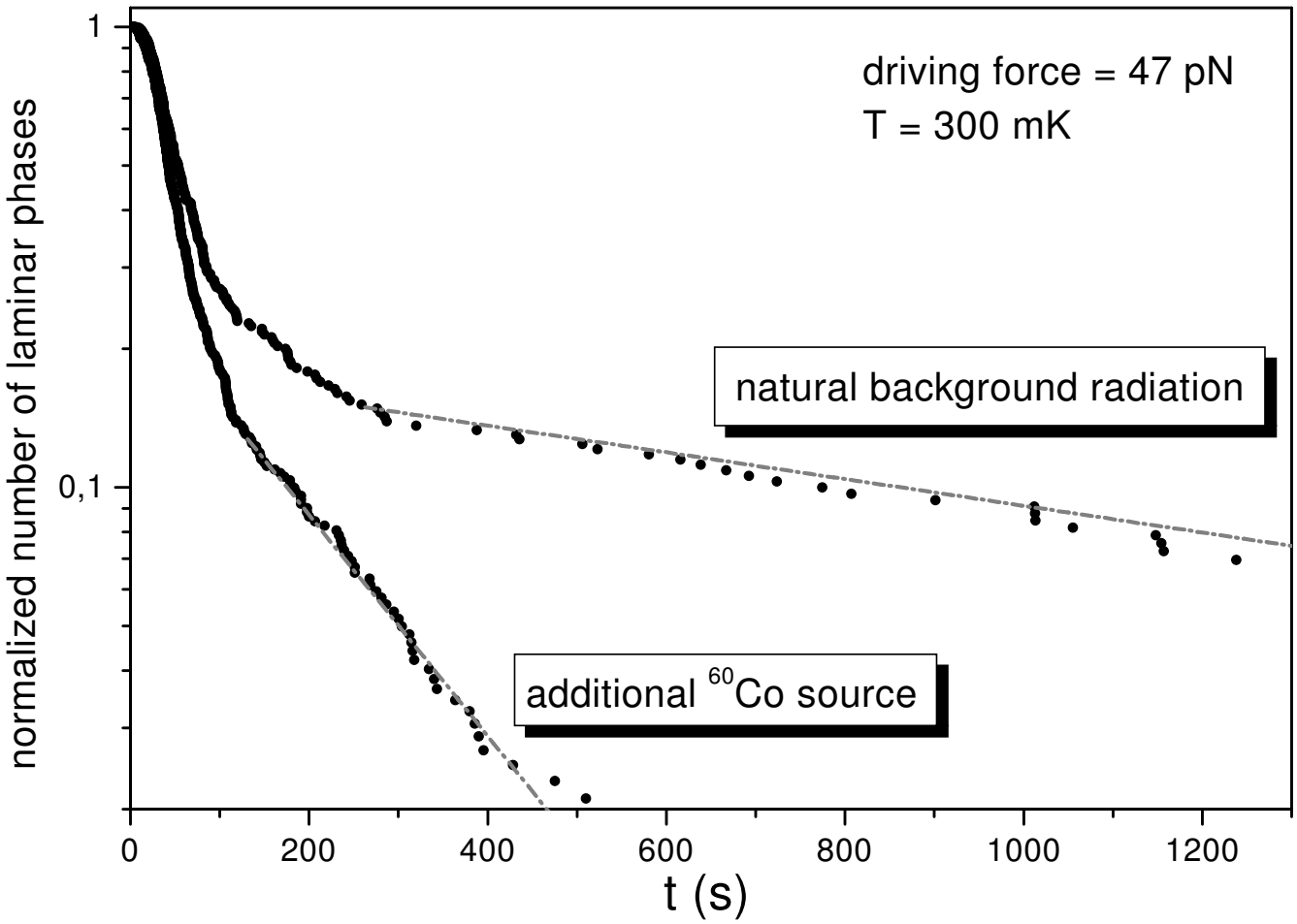
$\Lambda(t) = \Lambda(\Delta v)\ d\Delta v/dt = F_t(t)\,\Delta v(t)/m\,v_w^2 = P_t(t)/\frac{1}{2}mv_w^2$,

where $P_t(t)$ **is the power injected into the superfluid that is not dissipated by phonon scattering.**

$\Delta v(t) = \Delta v_{max}(1 - \exp(-t/\tau))$ (e.g.,τ = 31 s at 300 mK) and hence

for long times $t \gg \tau$: $\Delta v = \Delta v_{max} =$ const and therefore $\Lambda = 0$:

stable laminar phases although $v > v_c$


driving force = 47 pN
T = 300 mK
natural background radiation
additional 60Co source
normalized number of laminar phases
1
0,1
0
200
400
600
800
1000
1200
t (s)

Extreme-Value Statistics

$$m_n = \min\{X_1, \ldots\ldots\ldots, X_n\},$$

$$M_n = \max\{X_1, \ldots\ldots\ldots, X_n\}$$

The X_i are independent and identically distributed with CDF

$$F(x) = P(X_i \le x)$$

n is the sample size.

The distributions of the extrema are given by

$$P(M_n \le x) = F^n(x)$$

and $$P(m_n \le x) = 1 - (1 - F(x))^n$$

These results are exact. The question is: Are there distributions which are asymptotically approached as n grows?

The answer is given by Extreme-Value Theory:

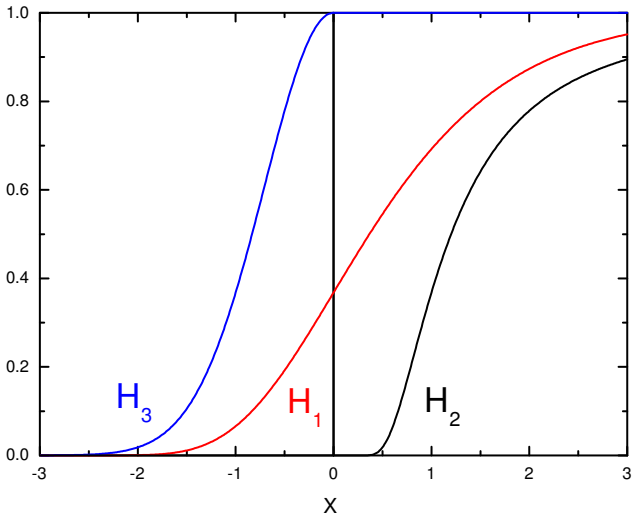
If a limit distribution exists it must belong to one of only three possible types.

The three extreme-value distributions for minima

$$\mathrm{H}_1(\mathrm{x}) = 1 - \exp\left(-\exp(\mathrm{x})\right),$$ **the "Gompertz" distribution**

$$\mathrm{H}_2(\mathrm{x}) = 1 - \exp\left(-(-\mathrm{x})^{-\alpha}\right), \ \mathrm{x} < 0, \ \alpha > 0,$$ **"Fréchet" distribution**

$$\mathrm{H}_3(\mathrm{x}) = 1 - \exp\left(-\mathrm{x}^{\alpha}\right), \ \mathrm{x} \geq 0, \ \alpha > 0,$$ **the "Weibull" distribution**


1.0
0.8
0.6
0.4
0.2
0.0
-3
-2
-1
0
1
2
3
X
H3
H1
H2

Domain of Attraction of Rayleigh Distribution

$$1 - \exp\left[-a_n \cdot (x - b_n)^2\right]$$

parent distribution:

	a_n	b_n
Gamma	$\frac{n}{2}$	0
log – logistic	$n - 1$	0
Beta	$\sim n$	0
Rayleigh	n	0

and therefore:

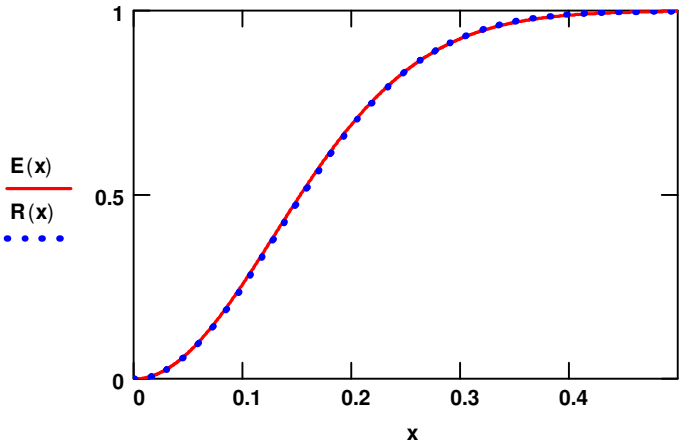
$$v_w \sim \frac{1}{\sqrt{n}}$$

1. Example: from log-logistic to Rayleigh

$$n = 30$$

$$F(x) = 1 - \frac{1}{1 + x^2} \qquad E(x) = 1 - (1 - F(x))^n$$

$$a = n - 1 \qquad b = 0 \qquad R(x) = 1 - \exp\left[-a \cdot (x - b)^2\right]$$

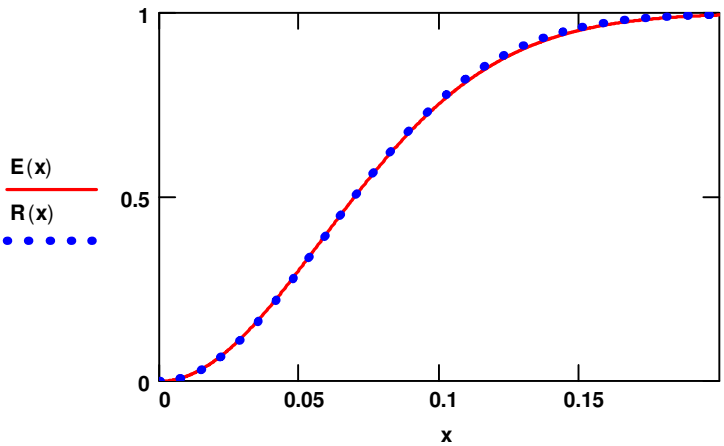


2. Example: from Gamma to Rayleigh

$$n = 300$$

$$F(x) = 1 - \exp(-x) \cdot (1 + x) \qquad E(x) = 1 - (1 - F(x))^n$$

$$a = n \cdot 0.48 \qquad b = 0 \qquad R(x) = 1 - \exp\left[-a \cdot (x - b)^2\right]$$



Summary

The measured parameters of the extreme value distribution can be evaluated quantitatively only if the parent distribution $F(x)$ and the sample size n are known.

Physical models for $F(x)$ are needed.

The effect of the geometry on n has to be investigated.