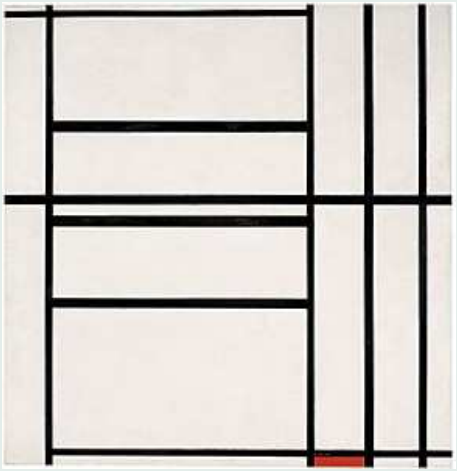
Vortex Network Generation in Superfluid Turbulence

–Defying Rotation–

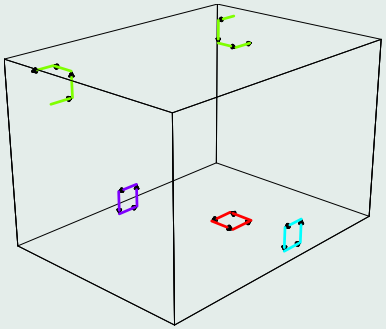
Adriaan M.J. Schakel

Institut für Theoretische Physik, Universität Leipzig

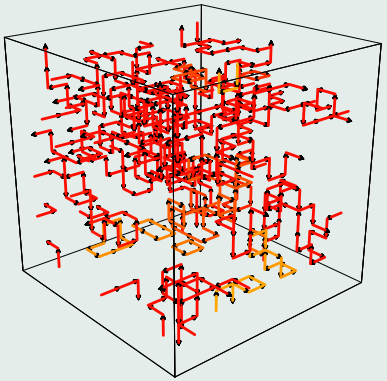


Piet Mondriaan, 1938-39

Vortices [Onsager, 1949], Monte Carlo [Krinner, Bittner, & Janke , 2003]:



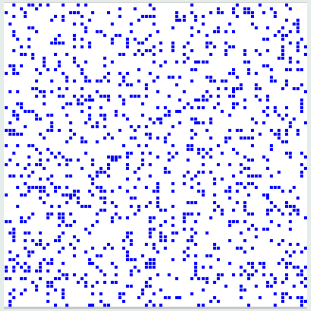
$T < T_c$



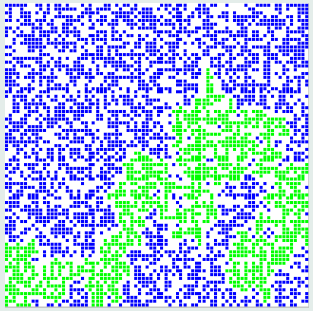
$T > T_c$

[Feynman, 1955]: *The superfluid is pierced through and through with vortex line. We are describing the disorder of Helium I.*

Cf. percolation:



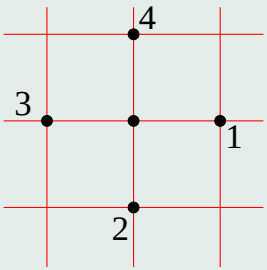
$p < p_c$



$p > p_c$

Definition (pseudocode):

```
while i < n
   generate m_i ∈ {1, 2, ···, 2d}
   move to n.n. m_i
   i++
end while
```

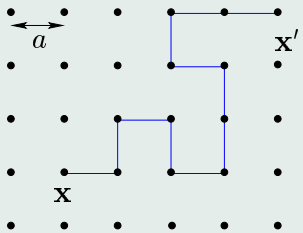


Probability:

$$K_n(\mathbf{x} \to \mathbf{x}') = \frac{\#\text{ of paths } \mathbf{x} \to \mathbf{x}' \text{ in } n \text{ steps}}{\#\text{ of paths } \mathbf{x} \to \ * \ \text{ in } n \text{ steps}}$$

Limit $n \to \infty$, $a \to 0$ such that $La = naa = \text{const}$:

$$\partial_n K_n(\mathbf{x} \to \mathbf{x}') = \frac{a^2}{2d} \nabla^2 K_n(\mathbf{x} \to \mathbf{x}')$$



Solution:

$$K_n(\mathbf{x} \to \mathbf{x}')/a^d = \left(\frac{d}{2\pi n a^2}\right)^{d/2} \exp\left[-\frac{d}{2}\frac{(\mathbf{x}-\mathbf{x}')^2}{na^2}\right]$$

Arbitrary vortex tangle:

$$\ln Z = \sum_{n=0}^{\infty} \frac{1}{n} K_n(0 \to 0) \times \mathrm{e}^{-\beta\theta n a} = \sum_n l_n$$

w/ vortex loop distribution:

$$l_n \propto n^{-\tau}\, \mathrm{e}^{-\beta\theta n a}$$

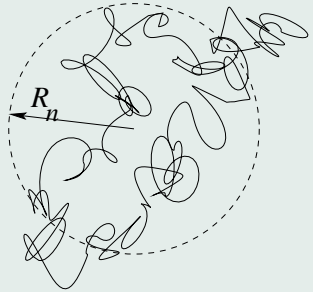
- Entropy factor w/ τ $(= d/2 + 1)$
- Boltzmann factor w/ line tension $\theta \propto |T_\mathrm{c} - T|^{1/\sigma}$ $(\sigma = 1)$: suppresses large loops

@ threshold $T = T_\mathrm{c}$, line tension vanishes:

$$l_n(T_\mathrm{c}) \propto n^{-\tau}$$ proliferation of loops

Cf. polymer physics:

$$R_n{}^2 = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})^2 \sim n^{2/D}$$



- $\bar{\mathbf{x}} = (1/n)\sum_{i=1}^{n}\mathbf{x}_i$
- D Hausdorff, or fractal dimension
  random walk: $D = 2$

R_n related to correlation length ξ:

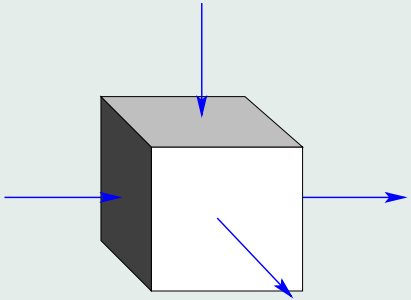
$$\xi \sim |T - T_c|^{-\nu} \quad \Rightarrow \quad \sigma = \frac{1}{\nu D} \quad \text{also} \quad \tau = \frac{d}{D} + 1$$

Scaling laws $\Rightarrow$ all critical exponents from vortex loop distribution

$$l_n(T) \propto n^{-\tau}\,\mathrm{e}^{-\beta\theta n a}, \quad \theta \propto |T_c - T|^{1/\sigma}$$

Problem w/ tracing out vortex tangle:

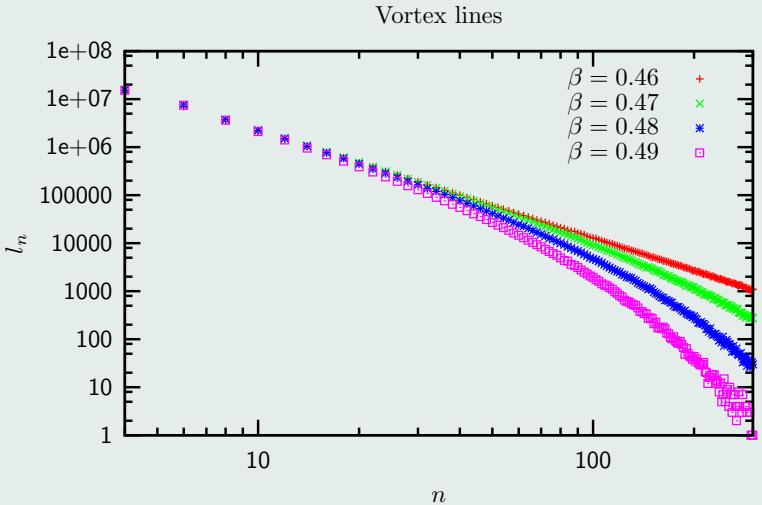
- randomly connect
- maximize vortex tangle



3DXY [Nguyen & Sudbø, 1997]:

$$1/\sigma = 1.45(5), \quad \tau = 2.4(1) \Rightarrow D > 2$$

Vortices: self-seeking



[Krinner, Bittner & Janke, 2004]

$\therefore$ Superfluid transition: Proliferation of vortex loops [Onsager, 1949]

2D Ising model in high-temperature representation ($J = 1$):

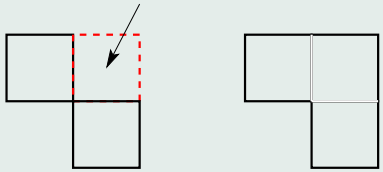
$$Z = (\cosh\beta)^{2N} 2^N \sum_{\text{Closed Graphs}} v^n, \qquad e^{\beta S_i S_j} = \cosh\beta(1 + v S_i S_j)$$

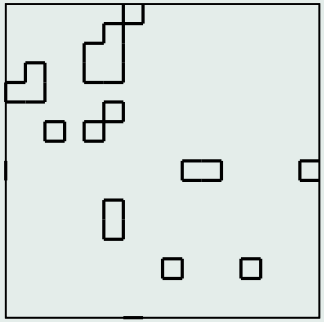
w/ $v = \tanh\beta$, n: # links in graph

Monte Carlo [Janke & A.S. (NPB to appear)]:

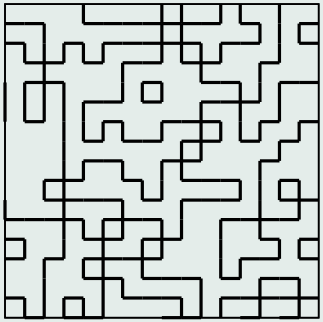
- Plaquette □ update
- Acceptance rate: $p_{\text{HT}} = \min\left(1, v^{n'-n}\right)$

selected plaquette



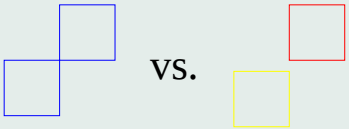


$\beta < \beta_c$



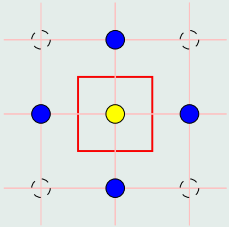
$\beta > \beta_c$

- Periodic boundary conditions
- Maximized tangles:

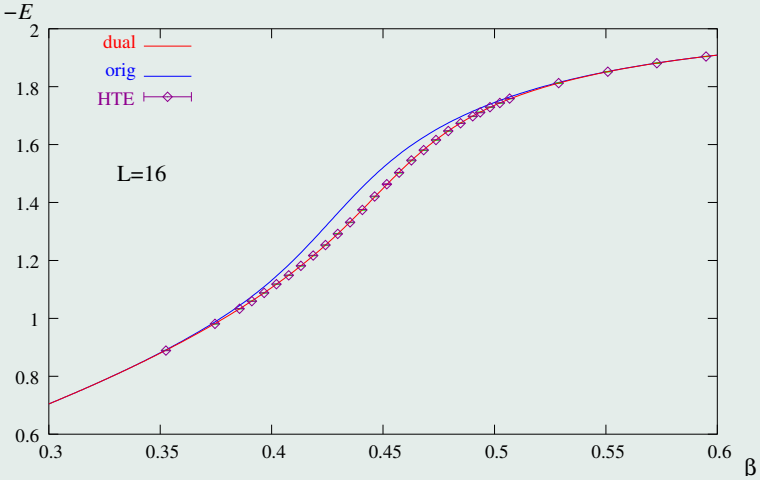


Phase transition: Proliferation of high-temperature graphs

Single plaquette □ update ~ Single spin update on dual lattice:

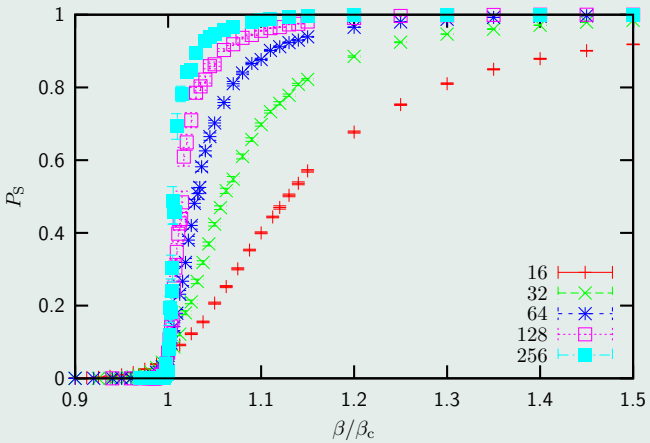


PBC: mismatch w/ $\tilde{\beta} = \frac{1}{2} \ln \coth \beta$ [Kramers & Wannier, 1941]:

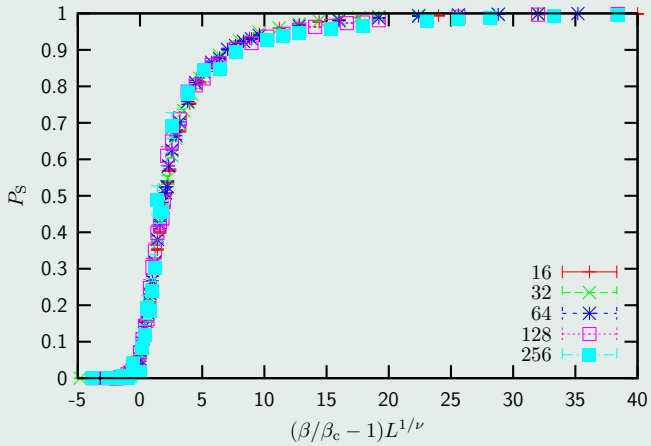


[Peierls, 1936]:

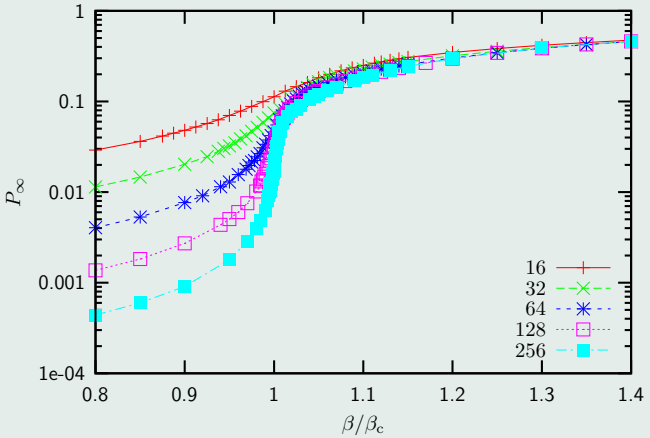
High-Temperature graphs ~ Boundaries of spin clusters (Domain walls)



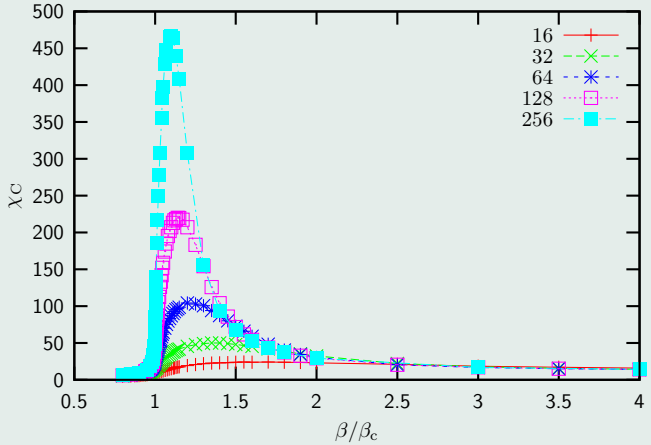
$\therefore$ Graph proliferation @ T_c



$\therefore \nu = 1$



$\therefore \beta_G = 0.627(8)$



$\therefore \gamma_G = 0.748(6)$

Closed graph distribution:

$$l_n(T) \propto n^{-\tau}\, \mathrm{e}^{-\theta n}, \quad \theta \propto |T_\mathrm{c} - T|^{1/\sigma}$$

w/

$$\sigma = \frac{8}{11}, \ \tau = \frac{27}{11}, \quad \Rightarrow \quad D_\mathrm{H} = \frac{11}{8}$$ [Duplantier & Saleur, 1988]

2D O(N) models:

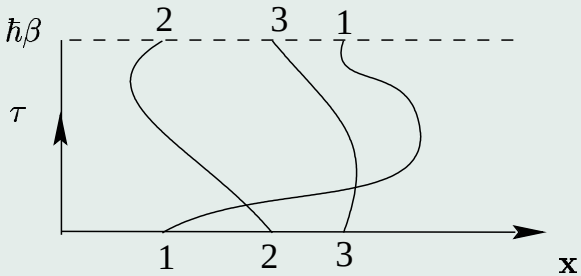
Model	N	c	γ	η	ν	D_H	σ
Gaussian	-2	-2	1	0	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{8}{5}$
SAW	0	0	$\frac{43}{32}$	$\frac{5}{24}$	$\frac{3}{4}$	$\frac{4}{3}$	1
Ising	1	$\frac{1}{2}$	$\frac{7}{4}$	$\frac{1}{4}$	1	$\frac{11}{8}$	$\frac{8}{11}$
XY	2	1	∞	$\frac{1}{4}$	∞	$\frac{3}{2}$	0

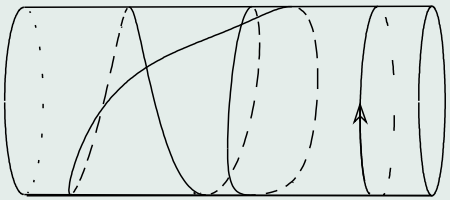
- XY-model ($N = 2$): $\sigma = 0$: algebraic behavior
- SAW ($N \to 0$) [de Gennes, 1972]: $\sigma = 1$: special
- In general: $\nu \neq 1/D_\mathrm{H}$, but

$$\nu = \frac{1}{\sigma D_\mathrm{H}}$$

Applied to:

- Percolation (Fortuin-Kasteleyn spin clusters)
- Bose-Einstein condensation: proliferation of Feynman's exchange rings





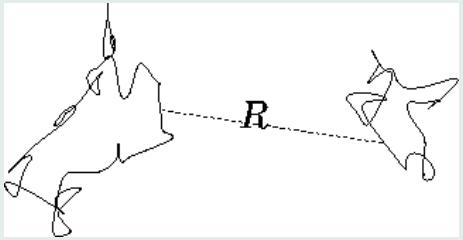
- Confinement: proliferation of center vortices (dual to Polyakov's loops)
- Surfaces (domain walls of 3D Ising model)
- Complex networks:
  - Internet
  - WWW
  - Coauthors
  - Citation statistics Phys. Rev.

"Action" of vortex lines $H = H_0 + H_{\rm int}$ w/ propertime parameter $s = \beta a^2 n/2d$:

$$H_0 = \sum_q \int_0^{s_q} \mathrm{d}s'_q \left[\tfrac{1}{4}\dot{\mathbf{x}}^2(s'_q) + \epsilon_L^2\right], \quad \epsilon_L^2 = \frac{2d}{\beta}\frac{\theta}{a} \ (>0)$$

Vortex tangle generates superflow ⇒ Biot-Sarvart interaction:

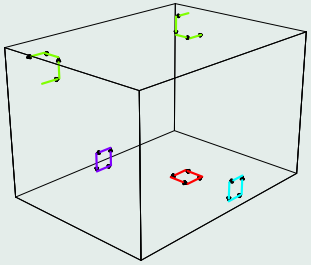
$$H_{\rm int} = \frac{1}{2}\frac{g^2}{4\pi}\sum_{q,q'}\int_0^{s_q}\mathrm{d}s'_q\int_0^{s_{q'}}\mathrm{d}s''_{q'}\,\dot{\mathbf{x}}(s'_q)\cdot\frac{1}{R}\,\dot{\mathbf{x}}(s''_{q'})$$



w/ $g^2 = \kappa^2\rho_{\rm s}$ (κ: circulation quantum)

Vortex loops ~ Worldlines of relativistic quantum particles

Vortex loop network:

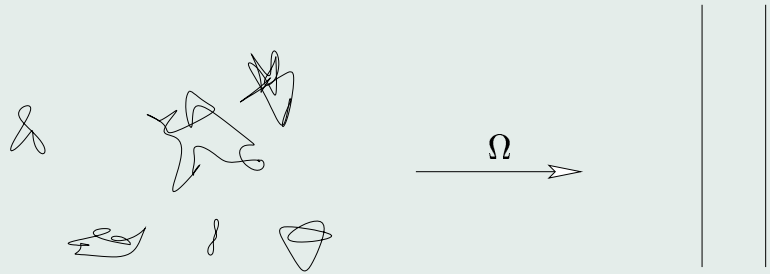


Feynman's path integral representation (note s)

$$Z = \sum_{N=0}^{\infty}\frac{1}{N!}\prod_{q=1}^{N}\left[\int_0^{\infty}\frac{\mathrm{d}s_q}{s_q}\oint\mathcal{D}\mathbf{x}(s'_q)\right]\mathrm{e}^{-\beta H}$$

Space (xy) & Time (z) on equal footing

Applying rotation Ω:



- Introduces preferred direction (z)
- Space (xy) & Time (z) become two independent structures
- Freezes in fluctuations in z (time) direction
- $\sim$ nonrelativistic limit, i.e., integral over propertime evaluated @ saddle point:

$$s = \frac{z}{2\epsilon_L}, \quad \epsilon_L : \text{ line tension}$$

- W/ $\mathbf{x}^2 = \mathbf{r}^2 + z^2$:

$$H_0 \xrightarrow{\text{n.r.}} \sum_q \int \mathrm{d}z \left[\tfrac{1}{2} \epsilon_L \dot{\mathbf{r}}_q^2(z) + \epsilon_L \right]$$

Assorted remarks:

- Superfluid turbulence (vortex tangle) natural from dual perspective
- Generation of superfluid turbulence in rotating superfluid through Kelvin wave instability:

Transition from nonrelativistic (order) to relativistic (disorder) vacuum and back

Open questions (stationary state):

- Relation with vortex tangle driving superfluid-normal phase transition?
- Vortex loop distribution?
- Fractal dimension?
- Different universality classes (grid/counterflow)?