

Vortex instability and the onset of superfluid turbulence

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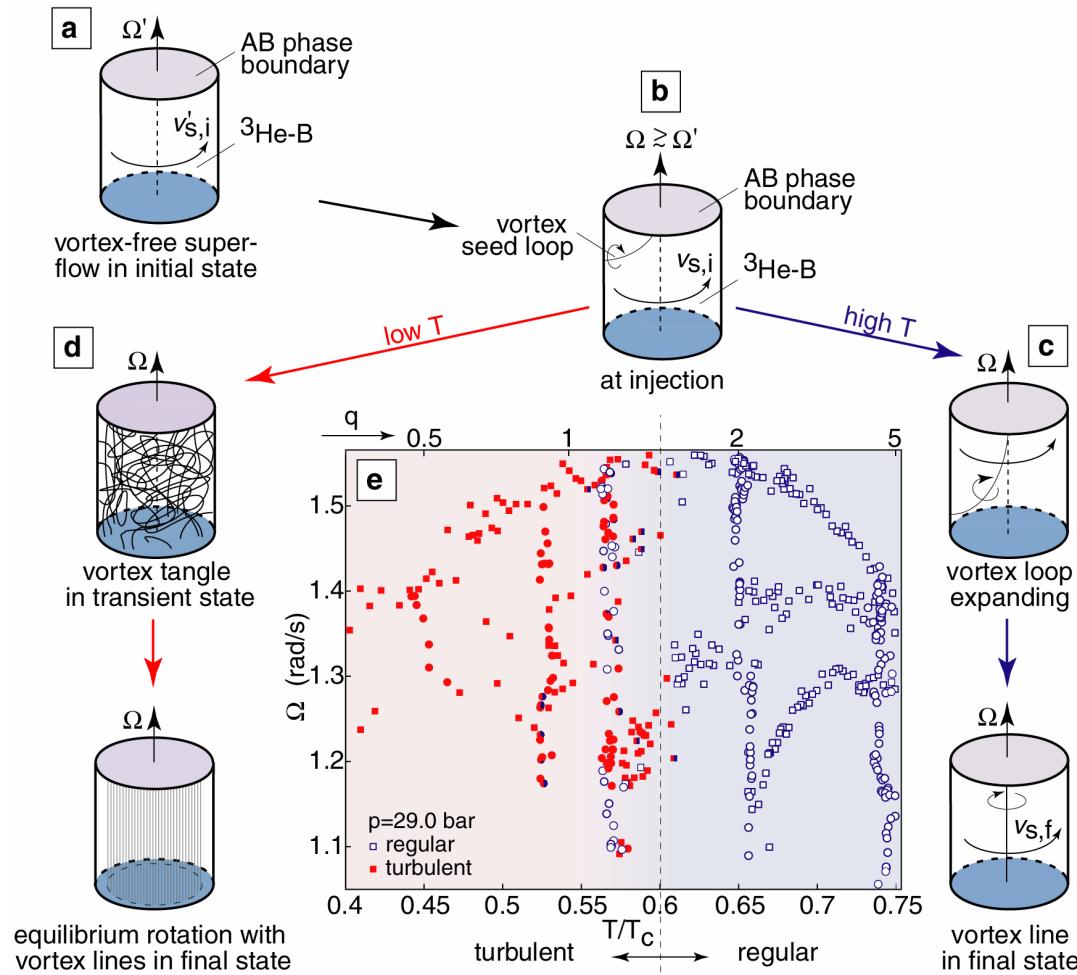
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- Experimental results on superfluid turbulence in He 3 B
 - New class of superfluid turbulence: Onset of turbulence independent of the Reynolds number.
- Results of numerical simulations
- Theoretical model for vortex instability
 - Competition between multiplication and removal of vortices.
 - Mutual-friction controlled onset of turbulence.

Superfluid turbulence. Results.

[Finne et al., Nature 424, 1022 (2002)]



Forces on vortices

- Magnus force

$$\mathbf{F}^M = \kappa \rho_s (\mathbf{v}_s - \mathbf{v}_L) \times \hat{\mathbf{z}}$$

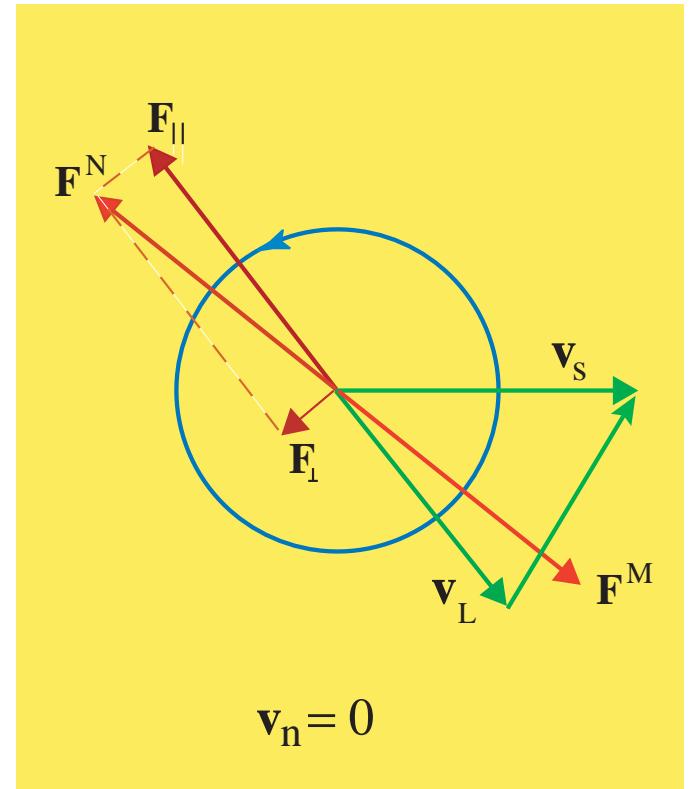
- Force from the normal component

$$\begin{aligned}\mathbf{F}^N = & \kappa \rho_s d (\mathbf{v}_n - \mathbf{v}_L) \\ & + \kappa \rho_s d' \hat{\mathbf{z}} \times (\mathbf{v}_n - \mathbf{v}_L)\end{aligned}$$

- Circulation quantum

$$\kappa = 2\pi\hbar/m^*$$

- Mutual friction parameters d, d'



Force balance for rectilinear vortex

$$\mathbf{F}^M + \mathbf{F}^N = 0$$

$$\mathbf{v}_L = \mathbf{v}_s + \alpha' (\mathbf{v}_n - \mathbf{v}_s) + \alpha \hat{\mathbf{z}} \times (\mathbf{v}_n - \mathbf{v}_s)$$

Hall and Vinen mutual friction parameters

$$\alpha = \frac{d}{d^2 + (1-d')^2}, \quad 1 - \alpha' = \frac{1-d'}{d^2 + (1-d')^2}$$

Important parameter

$$q = \frac{\alpha}{1 - \alpha'} = \frac{d}{1 - d'}$$

Mutual friction force on the superfluid

$$\mathbf{F}_{sn} = - \sum_{L_i} \mathbf{F}^M = -n_L \kappa \rho_s \alpha (\mathbf{v}_s - \mathbf{v}_n) + n_L \kappa \rho_s \alpha' [\hat{\mathbf{z}} \times (\mathbf{v}_s - \mathbf{v}_n)]$$

Numerical simulations

- The evolution is integrated from (Schwartz 1988)

$$\dot{\mathbf{s}} = \mathbf{v}_{\text{SI}} + \alpha' \mathbf{s}' \times [\mathbf{s}' \times \mathbf{v}_{\text{SI}}] - \alpha [\mathbf{s}' \times \mathbf{v}_{\text{SI}}]$$

A point on the vortex axis $\mathbf{s} = \mathbf{s}(\xi, t)$, where ξ is the arc length, $s' = \partial s / \partial \xi$, $\dot{\mathbf{s}} = d\mathbf{s}/dt = \mathbf{v}_L$.

The local superfluid velocity \mathbf{v}_{SI} includes all the Biot-Savart contributions

$$\mathbf{v}_{\text{SI}}(\mathbf{r}) = \sum_{L_i} \frac{\kappa}{4\pi} \int_{L_i} \frac{(\mathbf{s}_i - \mathbf{r}) \times d\mathbf{s}_i}{|\mathbf{s}_i - \mathbf{r}|^3}$$

The boundary conditions with image vortices are used.

- Vortex interconnections for crossing vortices.

Numerical results: High temperatures, high friction

Rotation velocity

$$\Omega = 0.21 \text{ rad/s}$$

“Superfluid Reynolds number”

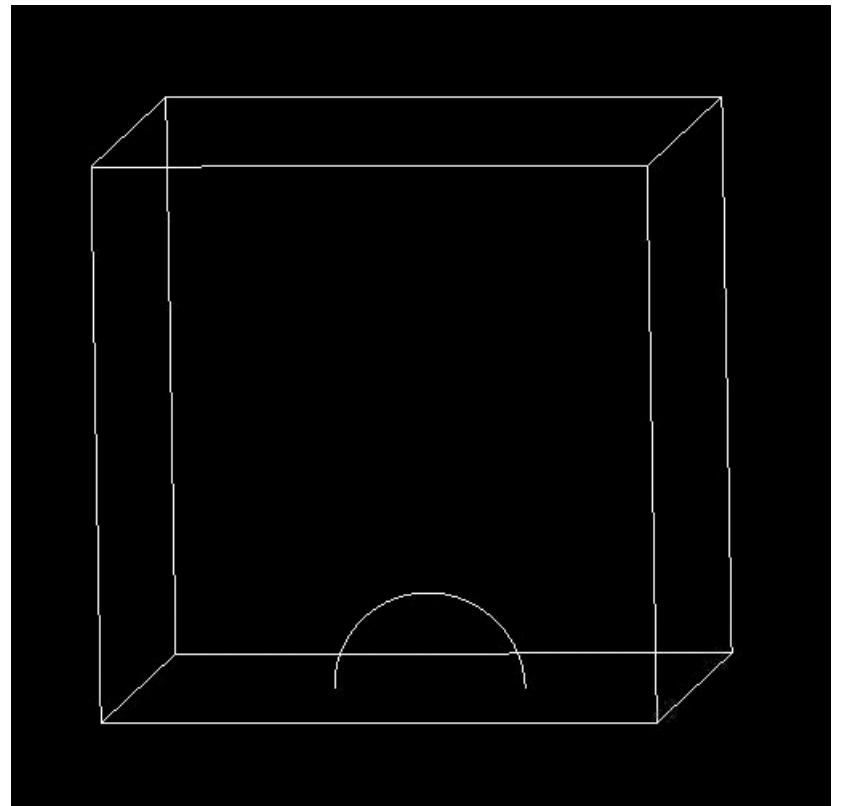
$$\text{Re}_s = \frac{U_s R}{\kappa} = 30$$

Temperature and the corresponding MF parameters

$$T = 0.8T_c, \quad q = 8.4$$

Evolution time

$$t_{\text{fin}} = 24 \text{ s}$$



Numerical results: Low temperatures, low friction

Rotation velocity and Reynolds number

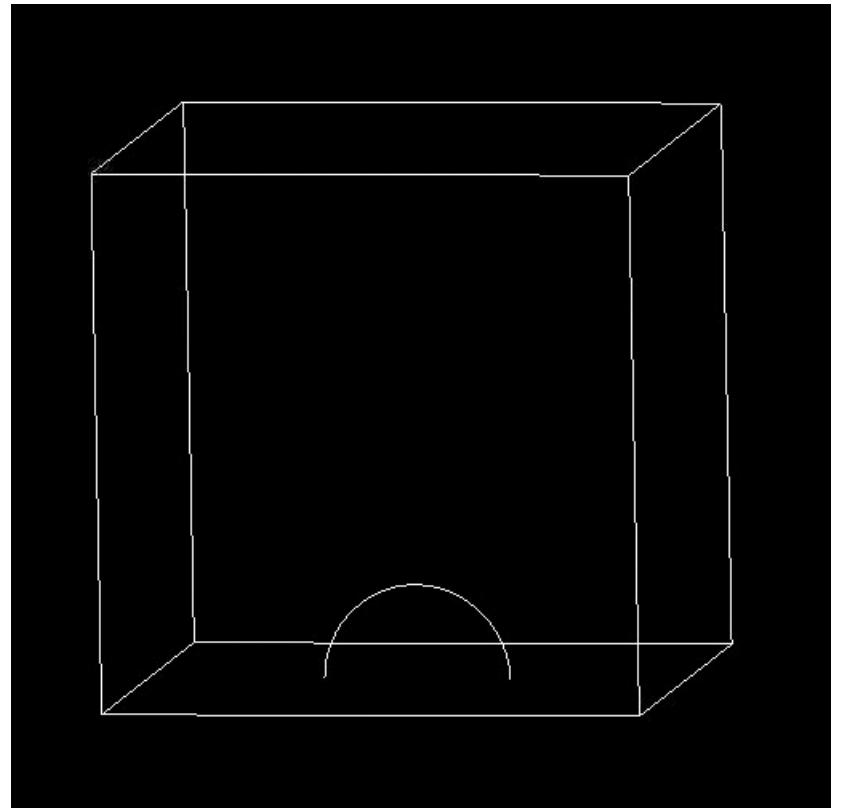
$$\Omega = 0.21 \text{ rad/s}, \text{Re}_s = 30$$

Temperature and MF parameters

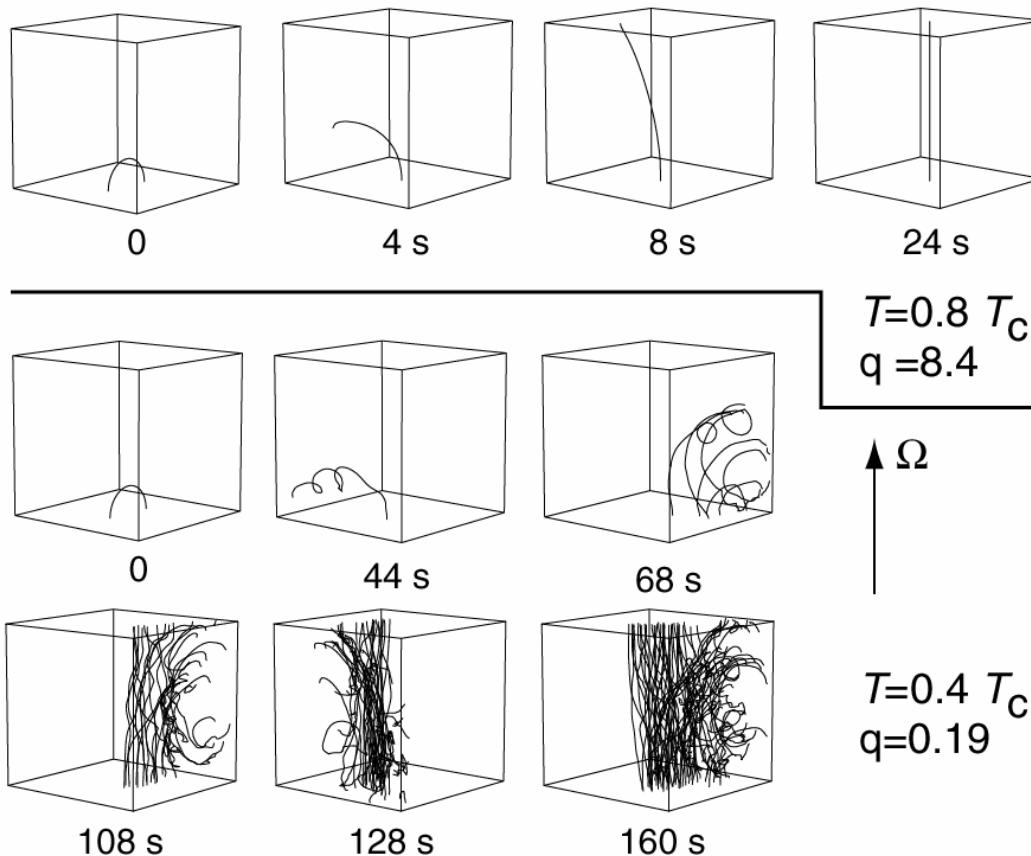
$$T = 0.4T_c, \quad q = 0.19$$

Evolution time

$$t_{\text{fin}} = 160 \text{ s}$$

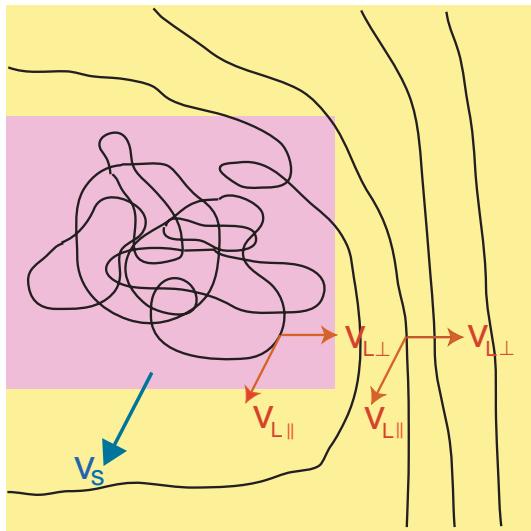


Numerical simulations of vortex evolution in a rotating container



Model for the onset of turbulence

[N.K., PRL **92**, 135301 (2004)]



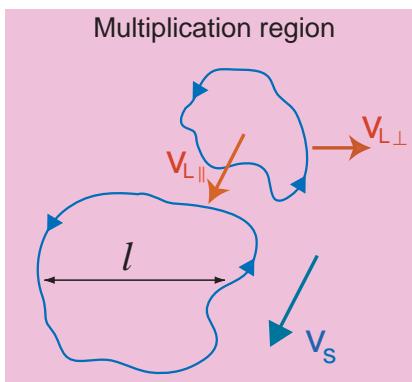
Vortex velocity

$$\mathbf{v}_L = (1 - \alpha')\mathbf{v}_s - \alpha \hat{\boldsymbol{\omega}}_s \times \mathbf{v}_s , \quad \boldsymbol{\omega}_s = \nabla \times \mathbf{v}_s$$

α and α' are the mutual friction parameters.

Competition between vortex multiplication and their extraction into the bulk.

$n \sim \ell^{-3}$ is the 3D vortex density, ℓ is a loop size,
 $L = \ell n \sim \ell^{-2}$ is the line density.



Multiplication due to collisions and reconnections

$$\dot{n}_+ = A v_{L\parallel} n^2 \ell^2 \Rightarrow \dot{L}_+ \sim (1 - \alpha') v_s L^{3/2}$$

Extraction due to inflation

$$\dot{\ell} \sim v_{L\perp} \sim \alpha v_s \Rightarrow \dot{L}_- \sim -\alpha v_s L^{3/2}$$

In total

$$\dot{L} = \dot{L}_+ + \dot{L}_- = \beta v_s L^{3/2}$$

Here

$$\beta = A(1 - \alpha') - B\alpha = (1 - \alpha')B(q_c - q),$$

$$q = \frac{\alpha}{1 - \alpha'}, \quad q_c = A/B \sim 1, \quad 1 - \alpha' > 0$$

The superfluid velocity $v_s \simeq U_s - v_0$.

The counterflow velocity U_s .

The self-induced velocity $v_0 \sim \kappa/\ell \sim \kappa L^{1/2}$.

Finally,

$$\frac{dL}{dt} = \beta [U_s L^{3/2} - \kappa L^2]$$

Similarly to the Vinen equation, (Vinen 1957).

Other approach: the vorticity equation

Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mu = \mathbf{v} \times \boldsymbol{\omega} + \nu \nabla^2 \mathbf{v}, \quad \boldsymbol{\omega} = \nabla \times \mathbf{v}$$

The vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times [\mathbf{v} \times \boldsymbol{\omega}] + \nu \nabla^2 \boldsymbol{\omega}$$

In superfluids: mutual friction force instead of viscosity

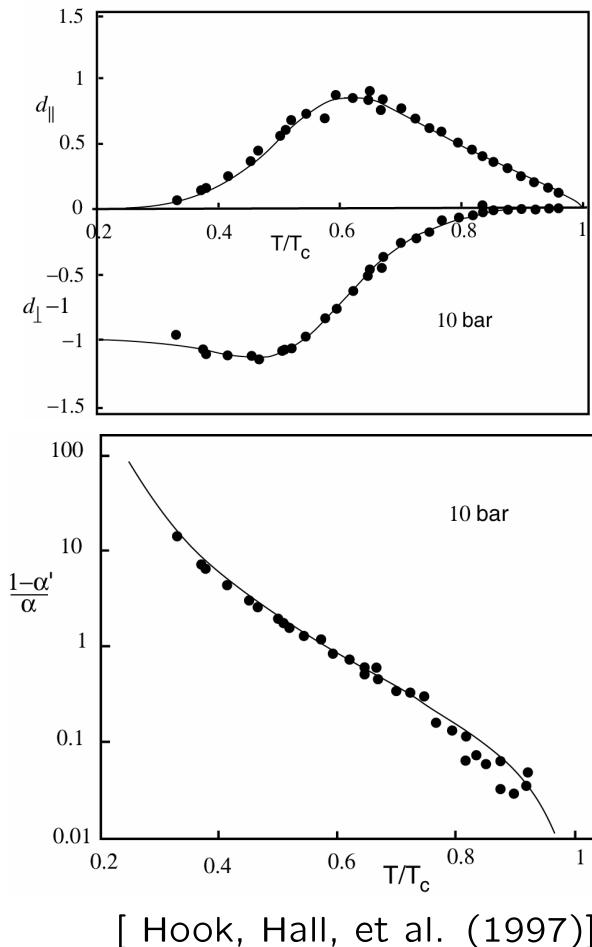
$$\mathbf{F}_{\text{sn}} = -\alpha' \rho_s [\mathbf{v}_s \times \boldsymbol{\omega}_s] + \alpha \rho_s [\hat{\boldsymbol{\omega}}_s \times [\boldsymbol{\omega}_s \times \mathbf{v}_s]]$$

$$\frac{\partial \boldsymbol{\omega}_s}{\partial t} = (1 - \alpha') \nabla \times [\mathbf{v}_s \times \boldsymbol{\omega}_s] + \alpha \nabla \times [\hat{\boldsymbol{\omega}}_s \times [\boldsymbol{\omega}_s \times \mathbf{v}_s]]$$

Average over random vortex loops assuming $\boldsymbol{\omega}_s \sim \kappa L$

$$dL/dt = \beta v_s L^{3/2} = \beta [U_s L^{3/2} - \kappa L^2]$$

Mutual friction parameters in He-3 B



Microscopic theory

[NK, Rep. Prog. Phys. (2002)]

$$\frac{1}{q} = \frac{1 - \alpha'}{\alpha} = \frac{1 - d_{\perp}}{d_{\parallel}} = \omega_0 \tau_{\text{eff}}$$

Distance between the CdGM bound states in the vortex core

$$\omega_0 \sim \Delta^2/E_F$$

Effective relaxation time

$$\tau_{\text{eff}}^{-1} \sim \tau_n^{-1}(T_c) \exp(-\Delta/T)$$

Thus

$$q^{-1} \sim \omega_0 \tau_{\text{eff}} \sim (\Delta/T_c)^2 \exp(\Delta/T)$$

Vortex instability

Finally

$$\frac{dL}{dt} = -\frac{\partial F}{\partial L}, \quad F = \beta \left[\frac{1}{3}\kappa L^3 - \frac{2}{5}U_s L^{5/2} \right]$$

Two regimes of evolution:

- $\beta > 0$, i.e., $q < q_c$, low T .

Stable solution

$$L_{max} \sim U_s^2 / \kappa^2 \gg L_\Omega = 2\Omega / \kappa$$

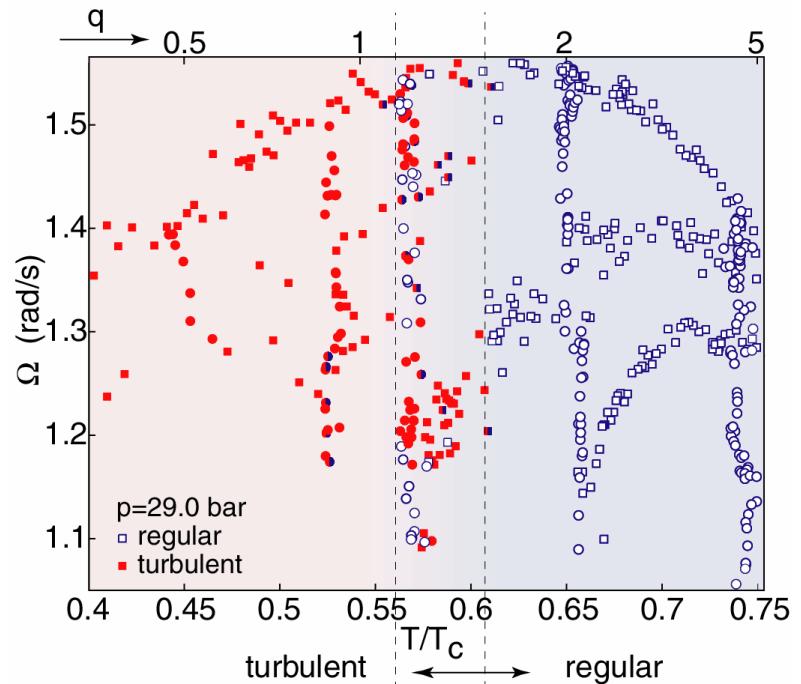
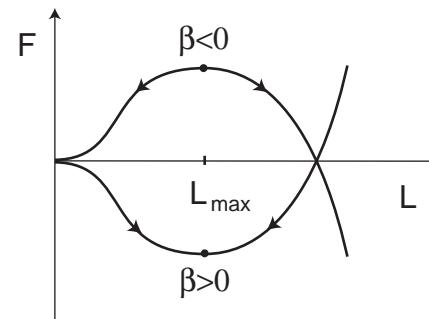
\Rightarrow Instability towards turbulent vortex tangle.

- $\beta < 0$, i.e., $q > q_c$, higher T .

Stable solution

$$L \rightarrow 0$$

\Rightarrow No multiplication.



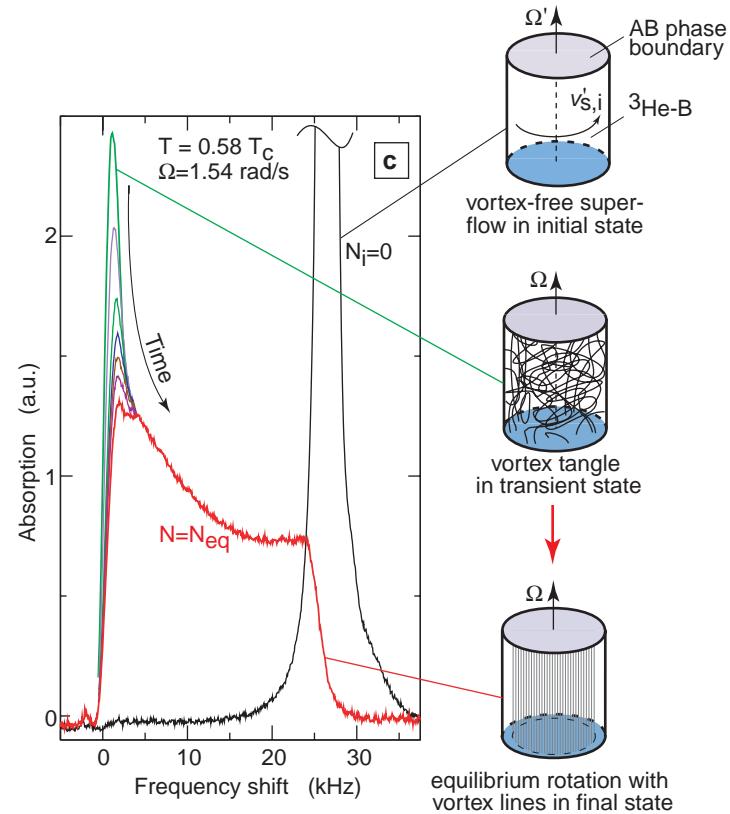
Superfluid turbulence in other systems

- He-3 A: High vortex friction; $q \gg 1$ except for very low temperatures $T \ll T_c$.
No turbulence.
- Superconductors: High vortex friction; $q \gg 1$ except for very clean materials, $l \gg (E_F/T_c)\xi$, and low temperatures.
No turbulence.
- Superfluid He II: Low vortex friction: $q \ll 1$ except for temperatures very close to T_λ .
Unstable towards turbulence.

Future projects in superfluid turbulence

Most challenging problems:

- Evolution and decay of developed superfluid turbulence.
- Interconnection between superfluid and classical turbulence



Summary

- Onset of superfluid turbulence is independent of velocity for $\text{Re}_s \gg 1$.
- Turbulence inducing (inertial) and stabilizing (dissipative) terms in the dynamic equation have the same scaling.
- Competition between inertial and dissipative terms is controlled by mutual friction.
- Low friction regime is turbulent while high friction regime is not.