ON MULTISCALE THEORY OF TURBULENCE IN WAVELET REPRESENTATION AND STOCHASTIC QUANTIZATION

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Wavelet based description of fluid flow and random fields

- Multistage description of hydrodynamic turbulence based on wavelet transform
- Multiscale random processes
- Connections to field theory
- Kolmogorov hypotheses revisited
- Multiscale stochastic quantization

$$\mathbf{u}_a(\mathbf{b},t) = \int \frac{1}{|a|^d} \bar{\psi}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right) \mathbf{u}(\mathbf{x},t)\, d^d\mathbf{x}$$

$$\langle W_a(b) W_{a'}(b') \rangle = D(a,b,a',b')$$

Down with UV divergences

Stochastic hydrodynamics

The stochastic hydrodynamics approach consists in introducing random force in the Navier-Stokes equations and calculating the velocity field moments using the stochastic perturbation theory . To exclude the pressure the Fourier transform is used.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\left(\frac{p}{\rho}\right) + \boldsymbol{\eta}(\mathbf{x},t), \left\langle \eta_i(x)\eta_j(x')\right\rangle = D_{ij}(x-x')$$

$$(-i\omega+\nu\mathbf{k}^2)u_i(k) - \int\frac{d^{d+1}q}{(2\pi)^{d+1}}M_{ijk}(\mathbf{k})u_j(q)u_k(k-q) = \eta_i(k),$$

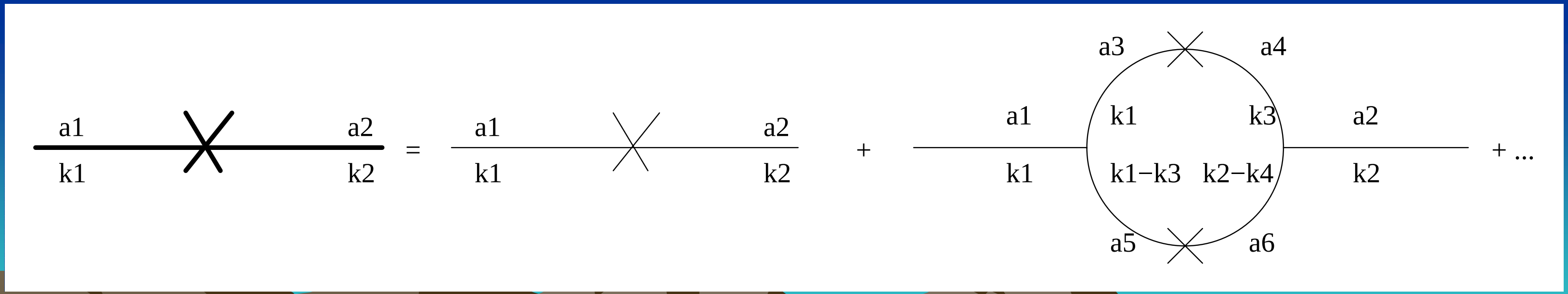
$$\left\langle \eta_i(k_1)\eta_j(k_2)\right\rangle = (2\pi)^{d+1}\delta^{d+1}(k_1+k_2)P_{ij}(\mathbf{k}_1)D(\mathbf{k}_1).$$

Diagram technique

Interaction vertex $$M_{ijk}(\mathbf{k}) = -\frac{i}{2}\left[k_j P_{ik}(\mathbf{k}) + k_k P_{ij}(\mathbf{k})\right],\quad P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}$$

Green function $$G(\omega,\mathbf{k}) = \frac{1}{-i\omega + \nu\mathbf{k}^2}$$

Force correlator: *D(ω,k)*



Continuous Wavelet Transform

For simplicity we restrict ourselves with the homogeneous isotropic media and real isotropic wavelets. In this case, the decomposition of the field into scale components is written as

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{C_\psi}\int_0^\infty \frac{da}{a^{d+1}}\int_{R^d} d^d\mathbf{b}\,\psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\mathbf{u}_a(\mathbf{b},t)$$

$$= \frac{1}{C_\psi}\int_0^\infty \frac{da}{a^{d+1}}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i(\mathbf{kx}-\omega t)}\hat{\psi}(a\mathbf{k})u_a(k),$$

$$\mathbf{u}_a(\mathbf{b},t) = \int \frac{1}{a^d}\overline{\psi}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\mathbf{u}(\mathbf{x},t)\,d^d\mathbf{x},\ u_a(k) = \overline{\hat{\psi}(a\mathbf{k})}u(k),$$

$$C_\psi = \int_0^\infty \frac{|\hat{\psi}(a\mathbf{k})|^2}{a}da < \infty$$

Scale-component equations

In the (a,k) representation the NSE for the incompressible fluid becomes a system of integro-differential equations for the scale components

$$(-i\omega+\nu\mathbf{k}^2)u_{ai}(k)=\eta_{ai}(k)$$

$$+\frac{1}{C_\psi^2}\int M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})u_{a_1j}(q)u_{a_2k}(k-q)\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}q}{(2\pi)^{d+1}},$$

$$M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})=\overline{\hat{\psi}(a\mathbf{k})}M_{ijk}(\mathbf{k})\hat{\psi}(a_1\mathbf{q})\hat{\psi}(a_2(\mathbf{k}-\mathbf{q}))$$

The closures and the perturbation theory are derived by substitution of the Fourier component fields *u(k)* by the scale components $u_a(k)$ and the integration over the measure $d\log(a)\,d^dk/(2\pi)^d$

What random force can fight loop divergences?

- The random force correlator should obey certain conditions to make the resulting theory physically feasible: (i) the energy injection by random force should be equal to the energy dissipation; (ii) the forcing should be localized at large scales.

$$\left\langle \eta_i(k_1)\eta_j(k_2)\right\rangle = (2\pi)^{d+1}\delta(k_1+k_2)P_{ij}(\mathbf{k}_1)D(\mathbf{k}_1)$$

The spectral density of the random force should have a suitable power-law behavior. In the simplest, but not very feasible physically, case of the white noise, delta-correlated in both space and time. **What is more realistic physically, is to have a random force concentrated in a limited domain in k-space . This case, however, is difficult to evaluate analytically in the standard perturbation theory. This can be done by wavelets.**

Multiscale random processes

- Let us take a white noise

$$\left\langle \eta(k_1)\eta(k_2)\right\rangle = 2\pi\delta(k_1+k_2)D_0$$

- Let us apply WT to it

$$\left\langle \eta_{a_1}(k_1)\eta_{a_2}(k_2)\right\rangle = 2\pi\delta(k_1+k_2)D_0\overline{\tilde{\psi}(a_1k_1)\tilde{\psi}(a_2k_2)}$$

- Trick: we can define another random process in (a,k) space that gives the same white noise in coordinate space!

$$\left\langle \eta_{a_1}(k_1)\eta_{a_2}(k_2)\right\rangle = 2\pi\delta(k_1+k_2)D_0C_\psi a_1\delta(a_1-a_2)$$

Conclusion: the space of multiscale random functions $u_a(x,\cdot)$ is just more general than that we are used to

$$f_A(x) = C_\psi^{-1}\int_A^\infty \psi\left(\frac{x-b}{a}\right)u_a(b)\frac{da\,db}{a^2}$$

Scale-dependent forcing in hydrodynamics

- Random force correlator

$$\left\langle \eta_{a_1 i}(k_1)\eta_{a_2 j}(k_2)\right\rangle = (2\pi)^{d+1}\delta^{d+1}(k_1+k_2)P_{ij}(\mathbf{k}_1)C_\psi a_1 \delta(a_1-a_2)D(a_1,\mathbf{k}_1)$$

Resulting equation after integration over all scales

$$u_{ai}(k) = G_0(k)\eta_{ai}(k) + \overline{\widetilde{\psi}(a\mathbf{k})}G_0^2(k)\lambda^2 \frac{4}{C_\psi}\int \frac{d^{d+1}q}{(2\pi)^{d+1}}\frac{da'}{a'}$$
$$\times M_{ijk}(\mathbf{k})\,|\,G_0(q)\,|^2\,\Delta_{jl}(\mathbf{q})G_0(k-q)M_{klm}(\mathbf{k}-\mathbf{q})$$
$$\times \widetilde{\psi}(a'\mathbf{k})\eta_{a'm}(k) + O(\lambda^4),$$

$$\Delta_{ij}(\mathbf{k}) = \frac{P_{ij}(\mathbf{k})}{C_\psi}\int \frac{da}{a}|\,\widetilde{\psi}(a\mathbf{k})|^2\, D(a,\mathbf{k})$$

$\hat{g}_n(\mathbf{k}) = (2\pi)^{d/2}(-i\,|\,\mathbf{k}\,|)^2 e^{-\mathbf{k}^2/2}$

Single-scale forcing

- $D(a,k)=D_0 a_0 \delta(a-a_0)$ and Mexican hat wavelet

$$\hat{g}_2(\mathbf{k}) = (2\pi)^{d/2}(-i\,|\,\mathbf{k}\,|)^2 e^{-\mathbf{k}^2/2}$$

$$C(k) = 2\lambda^2\,|\,G_0(k)\,|^2 \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \Delta(\mathbf{q})\Delta(\mathbf{k}-\mathbf{q}) \times c_2(\mathbf{k},\mathbf{q})\,|\,G_0(q)\,|^2\,|\,G_0(k-q)\,|^2$$

$$C^{d=3,n=2}(k \to 0) = \frac{7}{40}\,\frac{\mathbf{k}^2\,|\,G_0(k)\,|^2\,\pi^{3/2} a_0^3 D_0^2}{\nu^3\sqrt{2}}\,\lambda^2$$

Generalized Kolmogorov Hypotheses

- The Kolmogorov theory of the locally isotropic turbulence is formulated in terms of relative velocities $u(r,l)=u(r+l)-u(r)$.
- H1: For the locally isotropic turbulence with high enough *Re* the PDFs for the relative velocities are uniquely determined by ν and ε.
- H2: Under the same assumption as for H1 the turbulent flow is self-similar in small scales in the sense that

$$u(r,\lambda l) \xleftrightarrow{law} \lambda^h u(r,l), \lambda > 0.$$

$$h(x) = \begin{cases} +1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \end{cases}$$

H2g: Generalized Kolmogorov hyposesis Of self-similarity. Under the same assumption as for H1 the turbulent flow is self-similar in small scales in the sense, that the pulsations of the velocity defined as

$$u_l(b) = \int \frac{1}{l} \psi\left(\frac{x-b}{l}\right) u(x)\,dx$$

have the following power-law behavior

$|u_l(b)|^2 \sim l^{2h}$

$$\mathrm{Re} = \frac{u_l l}{\nu}$$

Energy flux to small scales

$$\Delta E_j = -\Delta t \bar{u}_s^{r\alpha} u_m^{l\beta} u_k^{j\alpha} \int d^d x \bar{\psi}_s^r(x) \psi_m^l(x) \frac{\partial \psi_k^j(x)}{\partial x^\beta}$$

For the orthogonal wavelets, that are used in numerical simulations, only the terms of coinciding scales r=l=j survive. The energy flux from the *j*-th scale to the next (*j+1*)-th scale is proportional to $|u_j|^3/(la^j)$ in exact accordance to the Kolmogorov phenomenological theory. In more general terms of nonorthogonal basic functions, there is a next term proportional to $u^{j+1} u^{j+1} u^j$. This term can be interpreted as material derivative $u^j \partial (u^{j+1})^2$ of the mean energy $(u^{j+1})^2/2$ of the small scale fluctuations traveling along the stream of large scale velocity u^j.

Stochastic quantization

- Let us consider a φ^3 model in Euclidean space
- The stochastic quantization consists in substitution of quantum fields in R^d by random fields in R^{d+1}: $\varphi(x)\rightarrow\varphi(x,\tau,\cdot)$ and calculating the Green functions by averaging over a random process governed by Langevin equation

$$S[\varphi(x)] = \int d^d x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{3!}\varphi^3\right]$$

$$\frac{\partial\varphi(x,\tau)}{\partial\tau} + \frac{\delta S}{\delta\varphi(x,\tau)} = \eta(x,\tau)$$

$$\langle \eta(x,\tau)\eta(x',\tau')\rangle = 2D_0\delta^d(x-x')\delta(\tau-\tau')$$

$$\langle \eta_{a_1 i}(k_1)\eta_{a_2 j}(k_2)\rangle = (2\pi)^{d+1}\delta^{d+1}(k_1+k_2)C_\psi a_1\delta(a_1-a_2)2D(a_1,\mathbf{k}_1)$$

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Thank You for your attention !